

LOGIC IN COMPUTER SCIENCE

INSTRUCTION: THIS IS AN OPEN BOOK AND OPEN NOTES TEST PRACTICE. ATTEMPT ALL QUESTIONS.

1. Applying Logic

Suppose you have two friends, Alice and Bob. You know that one of them drives a Chevrolet and the other drives a Dodge. Furthermore, one lives in El Paso, and the other in Fresno.

- (a) Using a language with constants a and b , and unary predicates C , D , E , and F , give a set of first-order sentences that models this situation as completely as possible.
- (b) Suppose you discover that the driver of the Chevrolet lives in Fresno and that Alice lives in El Paso. Who drives the Dodge? Justify your answer using resolution or by giving a deduction.

2. Proofs and Models

The completeness theorem for first-order logic tells us that each sentence either has a deduction (from \emptyset) or a counter-model (i.e., a model in which it is false). For each of the following sentences, either show there is a deduction or give a counter-model.

- (a) $\exists x \forall y (Qy \rightarrow \neg Qx)$
- (b) $\forall x \forall y (Pxy \rightarrow \exists z (Pxz \wedge Pzy))$
- (c) $\forall x (f(f(f(f(x)))) = x \rightarrow f(f(x)) = x \rightarrow f(x) = x)$
- (d) $\forall x (x = a \vee x = b) \rightarrow \forall y \forall z \forall w (y = z \vee y = w)$

3. Substructures

- (a) A formula is *universal* if it is of the form $\forall x_1 \cdots \forall x_n \theta$, where θ is quantifier-free. Show that if A is a substructure of B and $\models_B \phi[s]$ for some universal formula ϕ and variable assignment s , where s maps only to elements of A , then $\models_A \phi[s]$.
- (b) An \exists_2 formula is one of the form $\exists x_1 \cdots \exists x_n \theta$ where θ is universal. Show that if an \exists_2 sentence in a language with no function or constant symbols is true in a model A , then it is true in some finite substructure of A .

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4. Definability

Let Σ be a signature with equality, a single binary function symbol r , and no other function, constant, or predicate symbols. Let M be a Σ -model whose domain is the set of non-negative integers and that interprets r as follows: for $b \neq 0$, $r^M(a, b)$ is the remainder when a is divided by b . For $b = 0$, $r^M(a, b) = 0$.

- (a) Give a formula that defines $\{0\}$ in M .
- (b) Give a formula that defines $\{1\}$ in M .
- (c) Give a formula that defines the primes in M .
- (d) Give a formula that defines the even numbers in M .

5. Decidability

Suppose Σ is a signature with a countable number of symbols and T is a consistent, decidable, first-order Σ -theory.

- (a) Prove that if ϕ is any Σ -sentence, then $Cn(T \cup \{\phi\})$ is decidable.
- (b) Prove that there exists a consistent, decidable, *complete* first-order theory T' such that $T \subseteq T'$.