2.2 The principle of least action

We are now in a position to introduce a new fundamental principle that governs the dynamics of physical systems. The *principle of least action*, or *Hamilton's principle*, asserts that systems are characterized by a function $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$, called the *Lagrangian*. We write this more briefly as $L(\mathbf{q}, \dot{\mathbf{q}}, t)$. The Lagrangian function in turn determines the dynamics of the system (the *equations of motion*) as a calculus of variations problem.

To describe this, consider all smooth paths $\mathbf{q}(t)$ in $\mathcal{Q}$ with fixed boundary conditions at the endpoints. That is, we consider all smooth $\mathbf{q}(t)$ satisfying

$$\mathbf{q}(t_1) \;=\; \mathbf{q}^{(1)} \;, \qquad \mathbf{q}(t_2) \;=\; \mathbf{q}^{(2)} \;, \tag{2.5}$$

for some fixed time interval $[t_1, t_2]$, and with $\mathbf{q}^{(i)}$ fixed for $i = 1, 2$. Then the path followed by the system between these two positions is such that the *action*

$$S[\mathbf{q}(t)] \;\equiv\; \int_{t_1}^{t_2} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)\, \mathrm{d}t \;, \tag{2.6}$$

is *extremized*. The action S is a *functional* – that is, a function whose argument is itself a function – and the above extremal problem is of the same type encountered in the second year Calculus of Variations course.

That the Lagrangian depends only on $\mathbf{q}$ and $\dot{\mathbf{q}}$, but not higher derivatives, is related to the fact that one expects the dynamics of a system to be determined uniquely once one specifies $\mathbf{q}$ and $\dot{\mathbf{q}}$ at some initial time $t = t_0$. Compare to our discussion of Newton's second law in section 1.2. This expectation is itself sometimes elevated to a principle: the *Newton-Laplace determinacy principle*. It also means we expect the equations of motion to be second order differential equations for $\mathbf{q}(t)$.

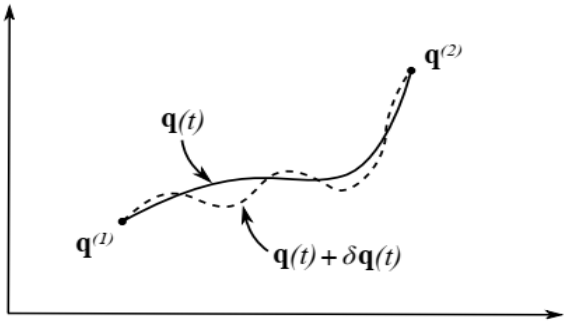

Figure 4: A trajectory $\mathbf{q}(t)$ in configuration space $\mathcal{Q}$, with fixed endpoints $\mathbf{q}(t_1) = \mathbf{q}^{(1)}$ and $\mathbf{q}(t_2) = \mathbf{q}^{(2)}$, together with a variation $\mathbf{q}(t) + \delta\mathbf{q}(t)$.

Let us determine the differential equations that result from extremizing the action S in (2.6). We suppose the extremum occurs at a critical function $\mathbf{q}(t)$, such that the change $\delta S = 0$ when $\mathbf{q}(t)$ is replaced by any variation $\mathbf{q}(t) \to \mathbf{q}(t) + \delta\mathbf{q}(t)$. Here $\delta\mathbf{q}(t)$ is a small change in $\mathbf{q}(t)$, where the fixed endpoints of the path require the boundary conditions $\delta\mathbf{q}(t_1) = \delta\mathbf{q}(t_2) = \mathbf{0}$. We may

make this more precise by writing

$$\delta\mathbf{q}(t) \;=\; \epsilon\,\mathbf{u}(t)\ , \tag{2.7}$$

$\mathbf{u}(t_1)=\mathbf{u}(t_2)=\mathbf{0}$, and ϵ is a real number. We then compute

$$\begin{aligned} S[\mathbf{q}(t)+\delta\mathbf{q}(t)] &= \int_{t_1}^{t_2} L(\mathbf{q}+\epsilon\,\mathbf{u},\dot{\mathbf{q}}+\epsilon\,\dot{\mathbf{u}},t)\,\mathrm{d}t \\ &= S[\mathbf{q}(t)]+\epsilon\sum_{a=1}^{n}\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_a}u_a+\frac{\partial L}{\partial\dot{q}_a}\dot{u}_a\right)\mathrm{d}t+O(\epsilon^2)\ . \end{aligned} \tag{2.8}$$

Here we have simply Taylor expanded around $\epsilon=0$ using the chain rule. Thus the first order variation in S is

$$\begin{aligned} \delta S &\equiv \epsilon\sum_{a=1}^{n}\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_a}u_a+\frac{\partial L}{\partial\dot{q}_a}\dot{u}_a\right)\mathrm{d}t \\ &= \epsilon\sum_{a=1}^{n}\left\{\int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q_a}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_a}\right)\right]u_a\,\mathrm{d}t+\left[\frac{\partial L}{\partial\dot{q}_a}u_a\right]_{t_1}^{t_2}\right\} \\ &= \epsilon\sum_{a=1}^{n}\int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q_a}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_a}\right)\right]u_a\,\mathrm{d}t\ , \end{aligned} \tag{2.9}$$

the last equality holding since $\mathbf{u}(t_1)=\mathbf{u}(t_2)=\mathbf{0}$. The requirement that $\mathbf{q}(t)$ is an extremum of S, $\delta S=0$, for *all* such variations $\delta\mathbf{q}(t)=\epsilon\,\mathbf{u}(t)$ then means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\mathbf{q}}}\right)-\frac{\partial L}{\partial\mathbf{q}} \;=\; \mathbf{0}\ , \tag{2.10}$$

or in components

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_a}\right)-\frac{\partial L}{\partial q_a} \;=\; 0\ ,\qquad a=1,\ldots,n\ , \tag{2.11}$$

The equations (2.10) are known in classical mechanics as *Lagrange's equations*, or the *Euler-Lagrange equations*. They are a set of n second-order differential equations for $\mathbf{q}(t)$. Notice that nothing we have said guarantees that the solution $\mathbf{q}(t)$ to (2.10) actually *minimizes* the action S. In fact there are examples where it does not *i.e.* the stationary point solution to $\delta S=0$ is not a minimum of the action It should thus more properly be called the *principle of stationary action.*

N.B. In deriving the Lagrange equations (2.10) one should be wary of the *first fundamental confusion of calculus* (N. M. J. Woodhouse, *Introduction to Analytical Dynamics*). The partial derivatives of L appearing in the second line of (2.8) involve regarding $L=L(\mathbf{q},\mathbf{v},t)$, where $\mathbf{v}=\dot{\mathbf{q}}$ is treated as an independent variable from $\mathbf{q}$. However, in computing the derivative $\mathrm{d}/\mathrm{d}t$ in the Lagrange equation (2.10) we mean the derivative of $\frac{\partial L}{\partial\mathbf{v}}(\mathbf{q}(t),\dot{\mathbf{q}}(t),t)$, which is a function only of time t. If you follow the derivation carefully these comments are self-evident, but it is also easy

to get confused! The potential confusion arises because we treat $\mathbf{q}$ and $\dot{\mathbf{q}}$ as independent variables on which L depend, but also $\mathbf{q} = \mathbf{q}(t)$ for the system trajectory.

As mentioned in the introduction, Lagrange's equations (2.10) hold in any coordinate system. This actually follows from the formulation of the principle of least action: it is *paths* that extremize S. If we change coordinates then the form of the path will also change, but only due to the coordinate transformation itself – it is the *same path* in $\mathcal{Q}$. One can check this statement very directly: if we make a coordinate transformation

$$\mathbf{q} \;=\; \mathbf{q}(\tilde{\mathbf{q}}, t) \;, \tag{2.12}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}_a}\right) - \frac{\partial \tilde{L}}{\partial \tilde{q}_a} \;=\; \sum_{b=1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_b}\right) - \frac{\partial L}{\partial q_b}\right]\frac{\partial q_b}{\partial \tilde{q}_a} \;, \qquad a = 1, \ldots, n \;. \tag{2.13}$$

Here we use the chain rule to compute

$$\dot{q}_a \;=\; \sum_{b=1}^{n}\frac{\partial q_a}{\partial \tilde{q}_b}\dot{\tilde{q}}_b + \frac{\partial q_a}{\partial t} \;=\; \dot{q}_a(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t) \;, \tag{2.14}$$

and define

$$\tilde{L}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t) \;\equiv\; L(\mathbf{q}(\tilde{\mathbf{q}}, t), \dot{\mathbf{q}}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t), t) \;. \tag{2.15}$$

That is, the transformed Lagrangian $\tilde{L}$ is obtained simply by substituting the coordinate transformation (2.12), (2.14) into the original Lagrangian. The Lagrangian is said to *transform as a scalar*. It is then common to drop the tilde on L, and simply remember that the notation $\partial L/\partial \tilde{\mathbf{q}}$ means one first substitutes $\mathbf{q} = \mathbf{q}(\tilde{\mathbf{q}}, t)$ to regard L as a function of $(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t)$, and then takes the partial derivative with respect to $\tilde{\mathbf{q}}$, with $\dot{\tilde{\mathbf{q}}}$ and t held fixed.

Going back to (2.13), since for a (non-singular) coordinate transformation the Jacobian matrix $\partial q_b/\partial \tilde{q}_a$ is invertible, we see that Lagrange equations (2.10) hold if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}}{\partial \dot{\tilde{\mathbf{q}}}}\right) - \frac{\partial \tilde{L}}{\partial \tilde{\mathbf{q}}} \;=\; \mathbf{0} \tag{2.16}$$

hold. Verifying (2.13) amounts to a short computation using the chain rule, and is left as an exercise on Problem Sheet 1.

Notice that if we have two Lagrangians L_1, L_2 related by

$$L_2(\mathbf{q}, \dot{\mathbf{q}}, t) \;=\; L_1(\mathbf{q}, \dot{\mathbf{q}}, t) + \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{q}, t) \;, \tag{2.17}$$

then these lead to the *same* Lagrange equations. This follows since the corresponding actions are related by

$$S_2 \;=\; S_1 + f(\mathbf{q}^{(2)}, t_2) - f(\mathbf{q}^{(1)}, t_1) \;, \tag{2.18}$$

where we have used the fundamental theorem of calculus. The actions thus differ by a term whose variation is zero. In classical mechanics (2.17) should be regarded as an equivalence relation on Lagrangians. Another comment is that if we have two Lagrangians L_1, L_2 describing *decoupled* systems, then the Lagrangian for the combined system is simply $L = L_1 + L_2$. This is reasonable, as then the equations of motion for both systems follow from that for L.

So far our comments have been very general. As mentioned in the introduction, the principle of least action (appropriately generalized) actually applies to many physical theories, including for example electromagnetism and general relativity. However, in classical mechanics the Lagrangian for a system takes the simple form

$$L = T - V \ , \tag{2.19}$$

where T is the kinetic energy of the system, and V is its potential energy. We shall see more precisely which systems this applies to in the next subsection, and why the Lagrange equations are equivalent to Newton's equations. Our application of Lagrangian mechanics to such systems will then always begin by identifying T and V, in whichever generalized coordinates we have chosen to use. Let us first see that this correctly reproduces Newton's second law as the Lagrange equations for a point particle:

Example (point particle in Cartesian coordinates): As in section 1.5 we consider a single point particle of mass m, in Cartesian coordinates $\mathbf{r} = (x_1, x_2, x_3)$ for an inertial frame $\mathcal{S}$. The particle is acted on by an external force $\mathbf{F} = -\nabla V$, where $V = V(\mathbf{r})$ is the potential. The Lagrangian (2.19) is thus

$$L = \frac{1}{2} m |\dot{\mathbf{r}}|^2 - V(\mathbf{r}) \ . \tag{2.20}$$

Here the generalized coordinates are $q_a = x_a$, $a = 1, 2, 3$. Since $\partial L/\partial \mathbf{q} = -\partial V/\partial \mathbf{r} = \mathbf{F}$ and $\partial L/\partial \dot{\mathbf{q}} = m\dot{\mathbf{r}} = \mathbf{p}$, we immediately deduce that the Lagrange equations (2.10) are simply Newton's second law. It is straightforward to extend this computation to a system of N particles with masses m_I at positions $\mathbf{r}_I(t)$, interacting through a potential $V = V(\mathbf{r}_1, \ldots, \mathbf{r}_N)$.

We thus see that Hamilton's principle (equivalently the Lagrange equations of motion) is equivalent to Newton's second law for point particles. Let's look at another simple example:

Example (simple pendulum): Let us return to the simple pendulum discussed in the previous subsection, and shown in Figure 2. Resolving the tension force $\mathbb{T}$ in the x and z directions, Newton's second law gives

$$m\ddot{x} = -\mathbb{T}\sin\theta \ , \qquad m\ddot{z} = -mg + \mathbb{T}\cos\theta \ . \tag{2.21}$$

Here recall $x = l\sin\theta$, $z = -l\cos\theta$. Taking $\cos\theta$ times the x equation of motion plus $\sin\theta$ times the z equation of motion we easily derive

$$\ddot{\theta} \;=\; -\frac{g}{l}\sin\theta \;, \tag{2.22}$$

which is the equation of motion for the coordinate θ. The other linear combination determines the tension

$$\mathbb{T} \;=\; mg\cos\theta + ml\dot{\theta}^2 \;, \tag{2.23}$$

which can be understood as balancing the component of the weight along the rod and the centripetal force for circular motion about the origin O, respectively.

On the other hand the Lagrangian description with generalized coordinate $q = \theta$ leads immediately to (2.22). The kinetic energy is $T = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) = \frac{1}{2}ml^2\dot{\theta}^2$, while the potential energy is $V = mgz = -mgl\cos\theta$. Thus the Lagrangian is

$$L \;=\; T - V \;=\; \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta \;, \tag{2.24}$$

and the Lagrange equation (2.10) with $q = \theta$ indeed gives (2.22). This short calculation might look like a sleight of hand: we have (apparently) entirely ignored the tension force, but still found the correct dynamics. In fact this is one of the advantages of the Lagrangian formalism, and we'll explain why (2.24) correctly describes the dynamics in the next subsection.

2.3 Constraints

In the simple pendulum example just studied the motion of the mass m in the (x, z) plane was constrained to satisfy $x^2 + z^2 = l^2$, due to the rod pivoted at O. This is an example of a *holonomic constraint*. To describe this more generally, we need some more notation. The constrained motion is by definition embedded inside a larger *unconstrained configuration space*. Let $x_1, \ldots, x_d$ denote (generalized) coordinates on this larger space, denoting $\mathbf{x} = (x_1, \ldots, x_d)$. For example, for the constrained motion of a single particle moving in $\mathbb{R}^3$ we may take $\mathbf{x} = (x, y, z)$ to be Cartesian coordinates. Then *holonomic constraints* on the motion constrain the coordinates to satisfy

$$f_A(\mathbf{x}, t) \;=\; 0 \;, \qquad A \;=\; 1, \ldots, d-n \;. \tag{2.25}$$

Here n is the number degrees of freedom satisfying the constraints. For example, for the simple pendulum the constraint on motion in the (x, z) plane is $f(x, z) = 0$ where $f(x, z) = x^2 + z^2 - l^2$. This is an example of a *scleronomous* constraint, meaning it doesn't depend on time t. More generally these are given by $f_A(\mathbf{x}) = 0$. The general time-dependent case in (2.25) are called *rheonomous* constraints, *e.g.* compare the bead moving on the fixed wire in Figure 3 to the case where the wire is rotating about the x-axis.

Let us fix the time t, and consider $f_A(\mathbf{x}, t) = 0$, $A = 1, \ldots, d-n$, which we assume are $d-n$ independent equations. As such, we expect to be able to eliminate $d-n$ of the d variables x_i, $i = 1, \ldots, d$, in terms of the remaining n variables. Equivalently we can say that there are functions

$$\mathbf{x} \;=\; \mathbf{x}(\mathbf{q}, t) \;, \tag{2.26}$$

which parametrize the general solution to (2.25). The variables q_a, $a = 1, \ldots, n$, are then generalized coordinates for the constrained motion. For example, for the simple pendulum we have $x = l\sin\theta$, $z = -l\cos\theta$, which solve the constraint $f(x, z) = 0$ where $f(x, z) = x^2 + z^2 - l^2$.

A precise definition of the constraints being *independent* is that the $(d-n) \times d$ matrix $\partial f_A / \partial x_i$ has maximal rank $d-n$ at every point. It then follows from the *implicit function theorem* that we can solve the constraints as in (2.26).[4] Moreover, there is always a system of coordinates $y_1, \ldots, y_d$ such that the constraints are simply

$$y_{n+A} \;=\; 0 \;, \qquad A \;=\; 1, \ldots, d-n \;. \tag{2.27}$$

We may then take $y_a = q_a$ for $a = 1, \ldots, n$ to be our generalized coordinates. Such a coordinate system is said to be *adapted* to the constraints $\{f_A = 0\}$. We will not give a proof of this result, but it follows from methods similar to those seen in the Part A course Introduction to Manifolds. As an example, the coordinates $y_1 = q_1 = \theta$, $y_2 = q_2 = \varphi$, $y_3 = r - a$ are adapted to the constraint $f(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0$ for motion on a sphere of radius a.

Going back to our dynamical problem, there will be a Lagrangian $L_0(\mathbf{x}, \dot{\mathbf{x}}, t)$ that describes the motion without the constraints imposed. For example, for a single particle of mass m moving in $\mathbb{R}^3$ under the influence of a potential V we have Cartesian coordinates $\mathbf{x} = (x_1, x_2, x_3) = (x, y, z)$ and this Lagrangian is $L_0 = L_0(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - V(x_1, x_2, x_3)$. Provided we can find generalized coordinates $\mathbf{q}$ that solve the constraints via $\mathbf{x} = \mathbf{x}(\mathbf{q}, t)$, as in (2.26), the Lagrangian for the constrained problem is simply

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) \;\equiv\; L_0(\mathbf{x}(\mathbf{q}, t), \dot{\mathbf{x}}(\mathbf{q}, \dot{\mathbf{q}}, t), t) \;. \tag{2.28}$$

In particular here

$$\dot{x}_i \;=\; \sum_{a=1}^{n} \frac{\partial x_i}{\partial q_a}\dot{q}_a + \frac{\partial x_i}{\partial t} \;, \qquad i = 1, \ldots, d \;, \tag{2.29}$$

by the chain rule. This is precisely what we did for the simple pendulum, for example computing the kinetic energy to be $T = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) = \frac{1}{2}ml^2\dot{\theta}^2$, where $x = l\sin\theta$, $z = -l\cos\theta$ solve the constraint in terms of the single generalized coordinate $q = \theta$. Solving the constrained problem in this way, the Lagrange equations for (2.28) are called *Lagrange equations of the second kind*.

[4] Those who have taken the Part A course Introduction to Manifolds may recognize the conditions we just stated.

Proof of Hamilton's principle for workless, holonomic constraints

In writing (2.28) we are stating that the Lagrangian for the constrained motion is obtained by simply substituting the solution to the constraints into the Lagrangian for the unconstrained motion. While this might seem reasonable, there is also an explanation. This is related to the *Lagrange equations of the first kind* where the constraints are not solved first, but are rather imposed via Lagrange multipliers. Specificially, let us introduce the Lagrangian

$$\hat{L} \;=\; \hat{L}(\mathbf{x},\dot{\mathbf{x}},\boldsymbol{\lambda},t) \;=\; L_0(\mathbf{x},\dot{\mathbf{x}},t) + \sum_{A=1}^{d-n} \lambda_A f_A(\mathbf{x},t)\ , \tag{2.30}$$

where the $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_{d-n})$ are called *Lagrange multipliers*, which are treated as additional generalized coordinates. The Lagrange equations for an extremum of the action $\hat{S} = \int_{t_1}^{t_2} \hat{L}\,\mathrm{d}t$ then read

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_0}{\partial \dot{x}_i}\right) - \frac{\partial L_0}{\partial x_i} \;=\; \sum_{A=1}^{d-n} \lambda_A \frac{\partial f_A}{\partial x_i}\ , \qquad i \;=\; 1,\ldots,d\ , \tag{2.31}$$

$$f_A(\mathbf{x},t) \;=\; 0\ , \qquad A \;=\; 1,\ldots,d-n\ . \tag{2.32}$$

Equation (2.31) is the Lagrange equation for x_i, while (2.32) follows from the Lagrange equation for λ_A: notice $\partial\hat{L}/\partial\lambda_A = f_A$, while $\hat{L}$ is independent of $\dot{\lambda}_A$.

We may now compare the equations (2.31), (2.32) to what we would obtain by applying Newton's laws. Of course (2.32) simply imposes the constraints. The left hand side of (2.31) is the unconstrained equation of motion for x_i. We have already shown that for an unconstrained system of point particles interacting via a general potential V, this is the same as Newton's second law. Concretely, for a single particle of mass m moving in $\mathbb{R}^3$ in a potential $V = V(\mathbf{x})$ (2.31) gives

$$m\ddot{x}_i + \frac{\partial V}{\partial x_i} \;=\; \sum_{A=1}^{d-n} \lambda_A \frac{\partial f_A}{\partial x_i} \;\equiv\; R_i\ , \qquad i = 1,\ldots,d\ . \tag{2.33}$$

The right hand side of (2.31)/(2.33) may then be interpreted as the (generalized) *constraint force* $\mathbf{R} = (R_1,\ldots,R_d)$. An example is the tension for the simple pendulum. By saying that a constraint force $\mathbf{R}$ is *workless* we mean that the work $\mathbf{R}\cdot\delta\mathbf{x} = 0$, where $\delta\mathbf{x}$ is any displacement tangent to the constraint space.[5] Geometrically this means that the vector $\mathbf{R}$ is orthogonal/normal to the constraint space. Since $\partial f_A/\partial x_i$ has maximal rank $d-n$, and for fixed A the vector field $\partial f_A/\partial\mathbf{x}$ is orthogonal to the constraint space $\{f_A = 0, A = 1,\ldots,d-n\}$, it follows that the set of $d-n$ vectors $\{\partial f_A/\partial\mathbf{x}, A = 1,\ldots,d-n\}$ form a basis for the normal space to the constraint space. Hence *any* constraint force $\mathbf{R}$ can be written as a linear combination $\mathbf{R} = \sum_{A=1}^{d-n}\lambda_A\frac{\partial f_A}{\partial\mathbf{x}}$. The Lagrangian (2.30) then determines the constrained dynamics, with the Lagrange multipliers determining the constraint forces.

[5] Some textbooks refer to such constraint forces as *ideal*, with $\mathbf{R}\cdot\delta\mathbf{x}$ the *virtual work* under the displacement $\delta\mathbf{x}$. It is "virtual" in the sense that $\delta\mathbf{x}$ isn't necessarily an actual displacement of the system obeying the equations of motion, but rather *any* potential displacement obeying the constraints.

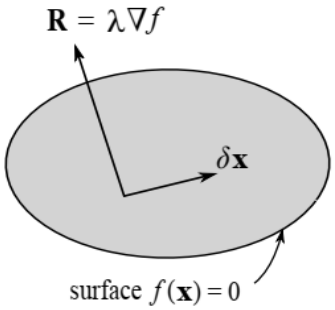

Figure 5: The figure shows a constraint surface defined by the single equation $f(\mathbf{x}) = 0$ in $\mathbb{R}^3$. A constraint force $\mathbf{R}$ is workless if $\mathbf{R} \cdot \delta\mathbf{x} = 0$ for any displacement vector $\delta\mathbf{x}$ tangent to the constraint space. Since $\nabla f = \partial f/\partial \mathbf{x}$ is everywhere normal to the constraint space we may hence write $\mathbf{R} = \lambda \nabla f$, where λ is the Lagrange multiplier.

That the Lagrange equations of the first and second kind are equivalent is then straightforward to see when we use coordinates $x_i = y_i$ that are adapted to the constraints. Recall that in this coordinate system the constraints are simply (2.27), with $y_a = q_a$, $a = 1, \ldots, n$, being generalized coordinates for the constrained motion. The constraints (2.32) then simply set $y_{n+A} = 0$ for each $A = 1, \ldots, d-n$, while the Lagrange equations (2.31) for $i = 1, \ldots, n$ read

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} \;=\; \sum_{A=1}^{d-n} \lambda_A \frac{\partial f_A}{\partial \mathbf{q}} \;=\; \mathbf{0}\,. \tag{2.34}$$

Here $L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$ is defined by (2.28), and the last equality $\partial f_A/\partial \mathbf{q} = \mathbf{0}$ holds since in this coordinate system the constraint functions $f_A = y_{n+A}$ are independent of $\mathbf{q}$. Notice that in this argument we have implicitly used the fact that the Lagrange equations hold in one coordinate system if and only if they hold in any other coordinate system. The remaining $d-n$ equations in (2.31) determine the Lagrange multipliers λ_A.

Example: Since the above discussion is a little abstract, let's return to our example of the simple pendulum. The unconstrained coordinates are $x_1 = x$, $x_2 = z$, while the single constraint function may be taken to be $f(x, z) = x^2 + z^2 - l^2$. The Lagrangian for the unconstrained motion is $L_0 = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) - mgz$, which is the Lagrangian for a particle of mass m moving in (x, z) space under gravity. The Lagrange equations of the first kind (2.31), (2.32) hence read

$$\begin{aligned} m\ddot{x} &= 2\lambda x\,, \\ m\ddot{z} + mg &= 2\lambda z\,, \\ x^2 + z^2 - l^2 &= 0\,. \end{aligned} \tag{2.35}$$

Here λ is the Lagrange multiplier. Polar coodinates (ϱ, θ), defined by $x = \varrho \sin\theta$, $z = -\varrho\cos\theta$, are essentially adapted to the constraint. More precisely, adapted coordinates may be taken to be

$y_1 = q = \theta$, $y_2 = \varrho^2 - l^2$, so the constraint is simply $y_2 = 0$. Imposing this of course sets $\varrho = l$, and the first two equations in (2.35) become

$$\begin{aligned} ml(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2) &= 2l\lambda\sin\theta\ , \\ ml(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2) + mg &= -2l\lambda\cos\theta\ . \end{aligned} \tag{2.36}$$

Taking $\cos\theta$ times the first equation plus $\sin\theta$ times the second equation then gives the equation of motion (2.22). We've already seen that this indeed arises as the Lagrange equation of the second kind for the system. On the other hand $\sin\theta$ times the first equation in (2.36) minus $\cos\theta$ times the second equation instead gives

$$-ml\dot{\theta}^2 - mg\cos\theta = 2l\lambda\ . \tag{2.37}$$

Comparing to (2.23) we thus see that the Lagrange multipler λ is indeed proportional to the constraint tension $\mathbb{T}$

$$\lambda = -\frac{1}{2l}\mathbb{T}\ . \tag{2.38}$$

The constraint force vector $\mathbf{R} = (2\lambda x, 2\lambda z) = (2l\lambda\sin\theta, -2l\lambda\cos\theta) = (-\mathbb{T}\sin\theta, \mathbb{T}\cos\theta)$ appears on the right hand side of (2.35), which is a vector of magnitude $\mathbb{T}$ that is normal to the constraint space.

It is also possible to have constrained motion where the constraints cannot be written in the form (2.25). These are called *non-holonomic constraints*. Examples include velocity dependent constraints $f(\mathbf{x}, \dot{\mathbf{x}}, t) = 0$, and constraints defined by inequalities, say $f(\mathbf{x}, t) \geq 0$. Velocity dependent constraints arise naturally in certain problems involving the rolling of rigid bodies, since the velocity of the point of contact is instantaneously zero. A particle confined to a box may be defined by inequalities on the coordinates. One can solve both types of problem, but the point is that our above treatment of holonomic constraints is not directly applicable. There is no general theory – the constraints are implemented on a case-by-case basis. However, it is perhaps worth noting that some velocity dependent constraints can be "integrated" to give equivalent holonomic constraints. For example consider the constraint $A(x,t)\dot{x} + B(x,t) = 0$. In general this would not be holonomic, but if there is a function $f(x,t)$ such that $A = \partial f/\partial x$ and $B = \partial f/\partial t$ then by the chain rule the constraint may be rewritten as $\mathrm{d}f/\mathrm{d}t = 0$, so that $f(x,t) =$ constant is an equivalent holonomic constraint (where we then need to also vary the constant in $f(x,t) =$ constant).