Classical Mechanics

LECTURE 4

2.4 Noether's theorem

The Lagrangian formulation of classical mechanics leads to a very clear relationship between *symmetries* and *conserved quantities*, via *Noether's theorem*. In this section we give a general account of Noether's theorem, and then discuss how certain symmetries lead to conservation of energy, momentum and angular momentum.

A *conserved quantity* is a function $F(\mathbf{q}, \dot{\mathbf{q}}, t)$ that is constant when evaluated on a solution to the Lagrange equations of motion, *i.e.*

$$0 \;=\; \frac{\mathrm{d}}{\mathrm{d}t}F(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) \;=\; \frac{\partial F}{\partial t} + \sum_{a=1}^{n}\left(\frac{\partial F}{\partial q_a}\dot{q}_a + \frac{\partial F}{\partial \dot{q}_a}\ddot{q}_a\right) , \tag{2.39}$$

where $\mathbf{q}(t)$ solves the Lagrange equations (2.10).

We next need to make precise what we mean by a *symmetry* of a Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$. There are various ways to approach this. We shall define a *generator of an infinitesimal deformation* to be a function $\boldsymbol{\rho} = \boldsymbol{\rho}(\mathbf{q}, \dot{\mathbf{q}}, t)$, which leads to a first order variation in any path $\mathbf{q}(t)$ in configuration space $\mathcal{Q}$ via

$$\delta\mathbf{q}(t) \;=\; \epsilon\boldsymbol{\rho}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) \;\equiv\; \epsilon\,\mathbf{u}(t) . \tag{2.40}$$

Notice here that $\mathbf{u}(t)$ depends on the path $\mathbf{q}(t)$, although for fixed path it is a function only of time t. Unlike the variations of the action in section 2.2 we do not require $\mathbf{u}(t)$ to satisfy any boundary conditions. Then we say that $\boldsymbol{\rho}$ generates a (infinitesimal) *symmetry* of L if there exists a function $f(\mathbf{q}, \dot{\mathbf{q}}, t)$ such that for all paths $\mathbf{q}(t)$ we have

$$\frac{\partial}{\partial\epsilon}L(\mathbf{q}(t) + \epsilon\,\mathbf{u}(t), \dot{\mathbf{q}}(t) + \epsilon\,\dot{\mathbf{u}}(t), t)\bigg|_{\epsilon=0} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}f(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) . \tag{2.41}$$

If one multiplies the left hand side of (2.41) by ϵ, this is the first order variation $\delta L \equiv \epsilon\left(\frac{\partial L}{\partial\epsilon}\big|_{\epsilon=0}\right)$ of L under $\mathbf{q} \to \mathbf{q} + \delta\mathbf{q}$. Thus (2.41) simply says that the first order variation in the Lagrangian is a total time derivative.

Noether's theorem: Suppose we have a symmetry of a Lagrangian L in the above sense. Then

$$F \;=\; F(\mathbf{q}, \dot{\mathbf{q}}, t) \;\equiv\; \sum_{a=1}^{n}\frac{\partial L}{\partial\dot{q}_a}\rho_a - f \tag{2.42}$$

is a *conserved quantity*.

The proof of Noether's theorem is by direct calculation. Evaluating on a solution $\mathbf{q}(t)$ to the Lagrange equations we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{a=1}^{n}\frac{\partial L}{\partial \dot{q}_a}\rho_a - f\right] &= \sum_{a=1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_a}\right)u_a + \frac{\partial L}{\partial \dot{q}_a}\dot{u}_a\right] - \frac{\mathrm{d}}{\mathrm{d}t}f(\mathbf{q}(t),\dot{\mathbf{q}}(t),t) \\ &= \sum_{a=1}^{n}\left[\frac{\partial L}{\partial q_a}u_a + \frac{\partial L}{\partial \dot{q}_a}\dot{u}_a\right] - \frac{\partial}{\partial \epsilon}L(\mathbf{q}(t)+\epsilon\mathbf{u}(t),\dot{\mathbf{q}}(t)+\epsilon\dot{\mathbf{u}}(t),t)\Big|_{\epsilon=0} \\ &= 0\ . \end{aligned} \tag{2.43}$$

Here in the second equality we have used the Lagrange equation on the first term, and (2.41) for the last term. The final equality simply follows from the chain rule.

The simplest case is when $f = 0$, so that the Lagrangian is actually invariant under the deformation to first order, $\delta L = 0$. In fact this is the version of Noether's theorem usually given in textbooks. The more general form we have presented will allow us to treat conservation of energy, momentum and angular momentum on the same footing. Notice that in deriving the conservation law (2.43) we only used the definition of the symmetry (2.41) for a solution $\mathbf{q}(t)$ to the equations of motion (rather than a completely general path).

Conservation of energy

Suppose we have a Lagrangian that is invariant under time translations, meaning that $L(\mathbf{q},\dot{\mathbf{q}},t+\epsilon) = L(\mathbf{q},\dot{\mathbf{q}},t)$ is independent of ϵ. Hence $\partial L/\partial t = 0$ and $\boldsymbol{\rho} = \dot{\mathbf{q}}$ generates a symmetry of L since

$$\frac{\partial}{\partial \epsilon}L(\mathbf{q}(t)+\epsilon\dot{\mathbf{q}}(t),\dot{\mathbf{q}}+\epsilon\ddot{\mathbf{q}}(t),t)\Big|_{\epsilon=0} = \frac{\mathrm{d}}{\mathrm{d}t}L(\mathbf{q}(t),\dot{\mathbf{q}}(t),t)\ . \tag{2.44}$$

The reasoning behind this is that the corresponding first order variation in a path arises from $\mathbf{q}(t+\epsilon) = \mathbf{q}(t) + \epsilon\dot{\mathbf{q}}(t) + O(\epsilon^2)$, so that $\delta\mathbf{q}(t) = \epsilon\dot{\mathbf{q}}(t)$. Thus from (2.44) we may simply take $f = L$ in Noether's theorem, and the conserved quantity is

$$H = \sum_{a=1}^{n}\frac{\partial L}{\partial \dot{q}_a}\dot{q}_a - L\ . \tag{2.45}$$

This is indeed what we would usually call the *energy*. To see this, let us suppose that $L = T - V$ with $V = V(\mathbf{q})$ and the kinetic energy being quadratic in the generalized velocities $\dot{\mathbf{q}}$

$$T = \frac{1}{2}\sum_{a,b=1}^{n} T_{ab}(\mathbf{q})\,\dot{q}_a\dot{q}_b\ . \tag{2.46}$$

For example this is the case if one is describing a point particle where the Cartesian coordinates x_a are related to the generalized coordinates q_a via the scleronomic change of variable $\mathbf{x} = \mathbf{x}(\mathbf{q})$. Then $T_{ab} = m\sum_{c=1}^{n}\frac{\partial x_c}{\partial q_a}\frac{\partial x_c}{\partial q_b}$. A simple computation then gives

$$H = 2T - L = T + V\ . \tag{2.47}$$

It is worth stressing that although H defined in (2.45) is always conserved when $\partial L/\partial t = 0$, it is not *always* the case that H is simply the sum of kinetic and potential energies $T + V$, which one would perhaps think of as the total energy. We might then have called this section "conservation of H", but what one defines as the "energy" is in any case a matter of convention.

We may also now define the *generalized momentum* $\mathbf{p}$ *conjugate* to $\mathbf{q}$ as

$$\mathbf{p} \equiv \frac{\partial L}{\partial \dot{\mathbf{q}}}\ . \tag{2.48}$$

For example, for a point particle in $\mathbb{R}^3$ this is simply the momentum $\mathbf{p} = m\dot{\mathbf{r}}$, where $\mathbf{r} = (x_1, x_2, x_3)$ are Cartesian coordinates. When $H = \sum_a p_a\dot{q}_a - L$ is regarded as a function of $(\mathbf{q}, \mathbf{p}, t)$, rather than $(\mathbf{q}, \dot{\mathbf{q}}, t)$, it is called the *Hamiltonian* of the system, and will be the subject of section 5. Notice that for a *closed* system the Lagrangian should indeed be time translation invariant, as this is one of the Galiliean symmetries discussed in sections 1.3 and 1.4. Conservation of energy follows directly from this symmetry.

Conservation of momentum

Suppose now that we have a *translational symmetry* of the configuation space, meaning that $\boldsymbol{\rho} = \mathbf{v}$ is a symmetry of the Lagrangian L with $f = 0$ and $\mathbf{v}$ a fixed vector. The corresponding first order variation in a path is $\delta\mathbf{q}(t) = \epsilon\mathbf{v}$ and $\mathbf{u}(t) = \mathbf{v}$. Since $\dot{\mathbf{u}} = \mathbf{0}$ the symmetry of the Lagrangian is equivalent to

$$\sum_{a=1}^{n} \frac{\partial L}{\partial q_a} v_a \;=\; 0 \; . \tag{2.49}$$

For example, for $\mathbf{v} = (1, 0, \ldots, 0)$ this means that $\partial L/\partial q_1 = 0$ and the coordinate q_1 is an *ignorable coordinate* (also sometimes called a *cyclic coordinate*). Noether's theorem immediately tells us that the quantity

$$\sum_{a=1}^{n} \frac{\partial L}{\partial \dot{q}_a} v_a \;=\; \sum_{a=1}^{n} p_a v_a \tag{2.50}$$

is conserved. Of course, this result may also be very simply derived by multiplying the Lagrange equations (2.10) by v_a and summing over a. Going back to our example with $\mathbf{v} = (1, 0, \ldots, 0)$ we see that it is the conjugate momentum p_1 that is conserved. Thus translational symmetry leads to conservation of momentum.

As an example consider the free motion of a particle in spherical polar coordinates $(q_1, q_2, q_3) = (r, \theta, \varphi)$, related to Cartesian coordinates by (2.4). Here the Lagrangian is $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\varphi}^2)$. Since $\partial L/\partial\varphi = 0$ we see that $\mathbf{v} = (0, 0, 1)$ generates a symmetry of L, and the conjugate momentum $p_3 = p_\varphi = \partial L/\partial\dot{\varphi} = mr^2\sin^2\theta\,\dot{\varphi}$ is conserved.

More generally for a closed system in an inertial frame we have the Galilean spatial translational symmetry discussed in sections 1.3, 1.4. Consider a system of N particles with masses m_I, position vectors $\mathbf{r}_I$, and interacting through a potential $V = V(\{|\mathbf{r}_I - \mathbf{r}_J|\})$. Here translational symmetry implies that the potential depends only on the distances between the particles $\{|\mathbf{r}_I - \mathbf{r}_J|\}$, and the Lagrangian

$$L \;=\; \frac{1}{2}\sum_{I=1}^{N} m_I |\dot{\mathbf{r}}_I|^2 - V(\{|\mathbf{r}_I - \mathbf{r}_J|\}) \tag{2.51}$$

is invariant under the spatial translations $\mathbf{r}_I \to \mathbf{r}_I + \epsilon\,\mathbf{k}$, for any ϵ and fixed vector $\mathbf{k}$. Here the configuration space has dimension $3N$ with generalized coordinates $(\mathbf{r}_1, \ldots, \mathbf{r}_N)$ and in the above notation the $3N$-vector is $\mathbf{v} = (\mathbf{k}, \ldots, \mathbf{k})$. Thus Noether's theorem gives us that

$$\sum_{I=1}^{N} \frac{\partial L}{\partial \dot{\mathbf{r}}_I} \cdot \mathbf{k} \;=\; \sum_{I=1}^{N} \mathbf{p}_I \cdot \mathbf{k} \tag{2.52}$$

is conserved. Since this is true for all $\mathbf{k}$ we deduce that the *total momentum*

$$\mathbf{P} \;=\; \sum_{I=1}^{N} \mathbf{p}_I \qquad (2.53)$$

is conserved. Compare this to our discussion in section 1.5.

Conservation of angular momentum

The Lagrangian (2.51) also has *rotational symmetry*. By this we mean that L is invariant under $\mathbf{r}_I \to \mathcal{R}\,\mathbf{r}_I$ for all $I = 1, \ldots, N$, where $\mathcal{R} \in O(3)$ is any rotation matrix. This is simply because L depends on $\mathbf{r}_I - \mathbf{r}_J$ and $\dot{\mathbf{r}}_I$ only via their lengths, which are (by definition) invariant under orthogonal transformations. We shall study rotations in more detail in section 4.1. Consider then a one-parameter family of rotations $\mathcal{R}(\epsilon) \in O(3)$, where $\mathcal{R}(0) = \mathbb{1}$ is the identity matrix. We may Taylor expand this in ϵ to write

$$\mathcal{R}(\epsilon) \;=\; \mathbb{1} + \epsilon \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} + O(\epsilon^2) \;, \qquad (2.54)$$

where $\mathbf{n}$ is a fixed vector describing the first order axis of rotation. Here we have simply used the fact that $\mathrm{d}\mathcal{R}/\mathrm{d}\epsilon \;|_{\epsilon=0}$ is an anti-symmetric matrix, which in turn follows from $\mathcal{R}(\epsilon)$ being orthogonal (see the discussion around equation (4.5)). Then $\mathbf{r}_I \to \mathcal{R}(\epsilon)\,\mathbf{r}_I$ is to first order in ϵ given by

$$\mathbf{r}_I \;\;\to\;\; \mathbf{r}_I + \epsilon\, \mathbf{n} \wedge \mathbf{r}_I \;. \qquad (2.55)$$

Invariance of the Lagrangian L in (2.51) gives, by Noether's theorem, that

$$\sum_{I=1}^{N} \frac{\partial L}{\partial \dot{\mathbf{r}}_I} \cdot (\mathbf{n} \wedge \mathbf{r}_I) \;=\; \sum_{I=1}^{N} \mathbf{n} \cdot (\mathbf{r}_I \wedge \mathbf{p}_I) \;=\; \mathbf{n} \cdot \mathbf{L} \qquad (2.56)$$

is conserved. Here we have used the scalar triple product identity in the second equality, and the definition of the *total angular momentum* $\mathbf{L}$ in (4.24) in the last equality. Since $\mathbf{n}$ is arbitrary we thus deduce that systems with rotational symmetry have conserved angular momentum. In comparing this to the more direct analysis in section 1.5 notice that the strong form of Newton's third law is obeyed by the Lagrangian (2.51), in the sense that for fixed I the force term $\partial L/\partial \mathbf{r}_I$ on the Ith particle is a sum of terms $\mathbf{F}_{IJ}$ for $J \neq I$, where $\mathbf{F}_{IJ} = -\mathbf{F}_{JI} \propto (\mathbf{r}_I - \mathbf{r}_J)$ follows since V depends only on $|\mathbf{r}_I - \mathbf{r}_J|$.

These are the most important symmetries and conservation laws in classical mechanics, but there are others. For example the Galilean boost symmetry of section 1.3 also leads to a conserved quantity, which again is not often mentioned in textbooks. The details are left to Problem Sheet 2.

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There are also conserved quantities that, while they can be derived from the form of Noether's theorem we have presented, are not obviously related to any geometric symmetry of the system. These are sometimes called *hidden symmetries*. The most famous example arises in the *Kepler problem* you studied in first year Dynamics. The Lagrangian is $L = \frac{1}{2}m|\dot{\mathbf{r}}|^2 + \frac{\kappa}{r}$, with κ a constant. This is the same potential as (1.11), and for example describes the motion of a planet around the Sun (more on this two-body problem in the next subsection). Then one can check explicitly that the *Laplace-Runge-Lenz vector*

$$\mathbf{A} \equiv \mathbf{p} \wedge \mathbf{L} - m\kappa \frac{\mathbf{r}}{|\mathbf{r}|} \tag{2.57}$$

is conserved, where $\mathbf{p} = m\dot{\mathbf{r}}$ and $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$ are the momentum and angular momentum about $\mathbf{r} = \mathbf{0}$, respectively. This is a fascinating conserved quantity, with a very interesting history. Writing $\mathbf{r} = (x_1, x_2, x_3)$, one can show that

$$\rho_a(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2}m\left[2\dot{x}_a x_b - x_a \dot{x}_b - \delta_{ab}\,\mathbf{r}\cdot\dot{\mathbf{r}}\right] , \tag{2.58}$$

where $b = 1, 2, 3$ is fixed generates a symmetry of the Lagrangian via $x_a(t) \to x_a(t) + \epsilon\,\rho_a(\mathbf{r}(t), \dot{\mathbf{r}}(t))$. The corresponding Noether conserved quantity in (2.42) (with $q_a = x_a$) is precisely the component A_b of the vector $\mathbf{A} = (A_1, A_2, A_3)$.

* Pauli used this vector to correctly derive the energy levels of the hydrogen atom *before* Schrödinger discovered his equation for quantum mechanics! Essentially the energy levels and their degeneracies follow from symmetry principles alone – one doesn't need to solve any differential equations. We'll come back to this topic again briefly in section 5. In the meantime there is more on the classical Laplace-Runge-Lenz vector (2.57) on Problem Sheet 1.

2.5 Examples

So far most of our discussion has been fairly abstract, sprinkled with a few very simple examples to hopefully clarify the basic ideas. However, one really gets to grips with Lagrangian mechanics by looking at various examples in detail. In this section we'll focus on the new ideas: finding generalized coordinates, deriving the Lagrangians, identifying symmetries and conserved quantities. Typically the Lagrange equations in any case cannot be solved in closed form (there are *chaotic* examples, such as the double pendulum). We will study systems analytically near to equilibrium in section 3, but more generally one will have to settle for a qualitative understanding and/or numerical simulations of the equations of motion.

Pendulum on a horizontally moving support

Consider a simple pendulum of mass m and length l, attached to a pivot of mass M that is free to move along the x-axis. As usual we denote by θ the angle the pendulum makes with the vertical z direction. If X denotes the position of the pivot mass M, then the pendulum mass m coordinates in the (x, z) plane are $(x, z) = (X + l\sin\theta, -l\cos\theta)$ – see Figure 6.

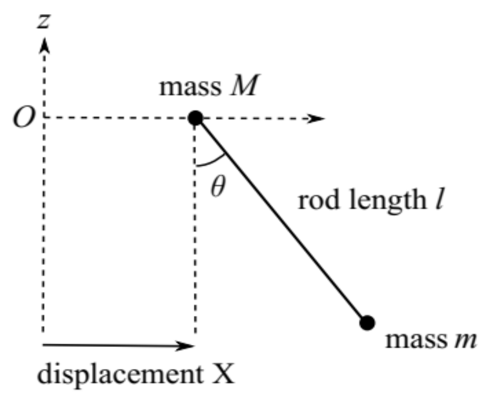

Figure 6: A pendulum on a horizontally moving support. The simple pendulum has mass m and length l, and the support pivot has mass M and moves freely along the x-axis with displacement X from the origin. The coordinates of the mass m are $(x, z) = (X + l\sin\theta, -l\cos\theta)$.

The total kinetic energy of the system is the sum of the kinetic energies of the two masses:

$$T \;=\; \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) \;=\; \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left[(\dot{X} + l\cos\theta\,\dot{\theta})^2 + (l\sin\theta\,\dot{\theta})^2\right] \;. \tag{2.59}$$

The potential energy is $V = mgz = -mgl\cos\theta$, precisely as it is for the usual simple pendulum. Gravity also acts on the pivot mass M, but since it is constrained to $z = 0$ this force is cancelled by a reaction force that we don't need to determine in the Lagrangian approach. Thus the Lagrangian is

$$L \;=\; T - V \;=\; \frac{1}{2}(M+m)\dot{X}^2 + ml\cos\theta\,\dot{X}\dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta \;. \tag{2.60}$$

Notice here that X is an *ignorable coordinate*, $\partial L/\partial X = 0$, so its conjugate momentum

$$p_X \;=\; \frac{\partial L}{\partial \dot{X}} \;=\; (M+m)\dot{X} + ml\cos\theta\,\dot{\theta} \tag{2.61}$$

is conserved (by Noether's theorem or by the Lagrange equation for X). The Lagrange equation of motion for θ is

$$0 \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\left[ml^2\dot{\theta} + ml\cos\theta\,\dot{X}\right] + ml\sin\theta\dot{X}\dot{\theta} + mgl\sin\theta \;. \tag{2.62}$$

The cross term in $\dot{X}\dot{\theta}$ cancels and this reduces to

$$\ddot{\theta} + \frac{1}{l}\cos\theta\,\ddot{X} + \frac{g}{l}\sin\theta \;=\; 0 \;. \tag{2.63}$$

One can then eliminate the $\ddot{X}$ term in favour of terms depending only on θ, $\dot{\theta}$, $\ddot{\theta}$ using $\dot{p}_X = 0$, thus obtaining a single second order ODE for $\theta(t)$:

$$(M + m\sin^2\theta)\,\ddot{\theta} + m\sin\theta\cos\theta\,\dot{\theta}^2 + \frac{g}{l}(M+m)\sin\theta \;=\; 0 \;. \tag{2.64}$$

However, since $\partial L/\partial t = 0$ we have another conserved quantity, namely the energy E. Since T is quadratic in the generalized velocities $\dot{X}$, $\dot{\theta}$ the argument after equation (2.46) applies and the conserved energy is hence

$$E \;=\; T + V \;=\; \frac{1}{2}(M+m)\dot{X}^2 + ml\cos\theta\,\dot{X}\dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta\ . \tag{2.65}$$

We may now substitute for the conserved generalized momentum p_X in (2.61) to obtain

$$E - \frac{1}{2(M+m)}p_X^2 \;=\; \frac{ml^2}{2(M+m)}(M + m\sin^2\theta)\dot{\theta}^2 - mgl\cos\theta\ . \tag{2.66}$$

We have thus reduced the problem to a *first order* ODE for $\theta(t)$. This was possible because of the existence of the two conserved quantities E and p_X. Equation (2.66) may be written as $\dot{\theta}^2 = f(\theta)$, which integrates to $t = \int \mathrm{d}\theta/\sqrt{f(\theta)}$. The solution $\theta(t)$ may then be substituted into the conserved momentum (2.61), which is a first order ODE for $X(t)$ which may similarly be directly integrated. We have thus *reduced the problem to quadratures*. Generally speaking this means writing the solution to n differential equations as a complete set of n independent definite integrals (generally using $n-1$ conservation laws, plus conservation of energy, with a final integration of the remaining variable after substituting for these conserved quantities). Such systems are also called *completely integrable*.

Bead on a circular wire/spherical pendulum

Next we discuss two closely related problems. Consider a small bead of mass m that slides freely on a circular wire of radius a which rotates about a vertical diameter. Here it is natural to introduce spherical polar coordinates (2.4) as these are adapted to the constraints. In particular the angle θ measures the position of the bead with respect to the vertical axis of rotation, while φ is the angle between the plane of the wire and some fixed fiducial (initial) choice of this plane – see Figure 7.

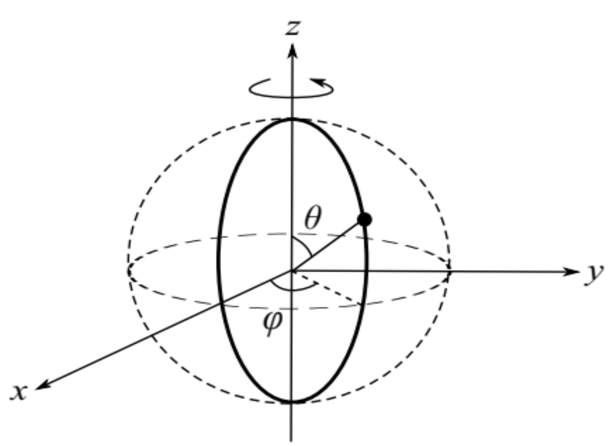

Figure 7: A bead of mass m sliding freely on a circular wire of radius a that rotates about a vertical diameter. One can use the spherical polar angles θ and φ as generalized coordinates.

The position of the bead in Cartesian coordinates is

$$\mathbf{r} = (x, y, z) = (a\sin\theta\cos\varphi, a\sin\theta\sin\varphi, a\cos\theta) . \tag{2.67}$$

One then calculates the kinetic energy

$$T = \frac{1}{2}m|\dot{\mathbf{r}}|^2 = \frac{1}{2}ma^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2) . \tag{2.68}$$

The potential energy is (notice that with respect to the simple pendulum we have $\theta \to \pi - \theta$)

$$V = mgz = mga\cos\theta . \tag{2.69}$$

We may then consider the following two dynamical situations:

1. The wire is forced to rotate at a constant angular velocity $\dot{\varphi} = \omega$.

2. The wire rotates freely about the vertical diameter.

Case 1 is an example of a time-dependent (rheonomous) holonomic constraint, since $\varphi(t)$ is fixed to be $\varphi(t) = \varphi_0 + \omega t$, which leaves just one degree of freedom in the angle θ. Case 2 on the other hand has two degrees of freedom θ, φ, and is equivalent to a *spherical pendulum* since the only constraint is that the bead is distance a from the origin. The two Lagrangians are hence

$$L_1 = L_1(\theta, \dot{\theta}) = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\omega^2\sin^2\theta - mga\cos\theta , \tag{2.70}$$

$$L_2 = L_2(\theta, \varphi, \dot{\theta}, \dot{\varphi}) = \frac{1}{2}ma^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2) - mga\cos\theta . \tag{2.71}$$

Focusing first on case 1, since $\partial L_1/\partial t = 0$ we may derive a first order equation from

$$E_1 = \frac{\partial L_1}{\partial \dot{\theta}}\dot{\theta} - L_1 = \frac{1}{2}ma^2(\dot{\theta}^2 - \omega^2\sin^2\theta) + mga\cos\theta \tag{2.72}$$

being conserved. Notice this is *not* the same as taking $T + V$ given by (2.68), (2.69) and putting $\dot{\varphi} = \omega$. This gives something very similar to (2.72), but with a minus sign difference (and is not conserved). Notice also that in (2.70) the kinetic term arising from $\frac{1}{2}ma^2\sin^2\theta\,\dot{\varphi}^2$ with $\dot{\varphi} = \omega$ is absorbed into an effective potential

$$V_{\text{eff}}(\theta) = mga\cos\theta - \frac{1}{2}ma^2\omega^2\sin^2\theta \tag{2.73}$$

for the dynamics of $\theta(t)$. We may again in principle integrate the equation $E_1 = \frac{1}{2}ma^2\dot{\theta}^2 + V_{\text{eff}}(\theta)$, thus reducing to quadratures. The equation of motion for θ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_1}{\partial \dot{\theta}}\right) = \frac{\partial L_1}{\partial \theta} \qquad \Longleftrightarrow \qquad ma^2\ddot{\theta} = -\frac{\partial V_{\text{eff}}}{\partial \theta} = ma\sin\theta(g + a\omega^2\cos\theta) , \tag{2.74}$$

and there are hence equilibrium points (see section 3) at $\theta = 0$ and $\theta = \pi$, so that the bead is on the axis of rotation, and provided $\omega \geq \sqrt{g/a}$ also at

$$\theta_0 = \cos^{-1}\left(-\frac{g}{\omega^2 a}\right) . \tag{2.75}$$

The stability of these equilibria is examined in Problem Sheet 2 (using the methods of section 3).

Going back to case 2, which is the spherical pendulum, the coordinate φ is ignorable, so its conjugate momentum

$$p_\varphi = \frac{\partial L_2}{\partial \dot{\varphi}} = ma^2 \sin^2\theta\,\dot{\varphi} \tag{2.76}$$

is conserved. This is simply the component of angular momentum about the vertical axis. Again $\partial L_2/\partial t = 0$ so

$$\begin{aligned} E_2 &= \frac{\partial L_2}{\partial \dot{\theta}}\dot{\theta} + \frac{\partial L_2}{\partial \dot{\varphi}}\dot{\varphi} - L_2 \\ &= \frac{1}{2}ma^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) + mga\cos\theta \end{aligned} \tag{2.77}$$

is conserved. This is simply $T + V$ for the bead. One can eliminate $\dot{\varphi}$ using (2.76), thus again reducing to quadratures.