Classical Mechanics

The double Atwood machine

Next we consider the *double Atwood machine*, shown in Figure 8. This consists of a cable of length l passing over a *fixed* pulley, with a mass m_1 tied to one end of the cable and another *moveable* pulley tied to the other end. Over this second pulley passes a cable of length l', with masses m_2, m_3 tied to each end. We treat the pulleys and cables as massless, there is no friction, and moreover we take the radii of the pulleys to be negligible compared to l, l'.

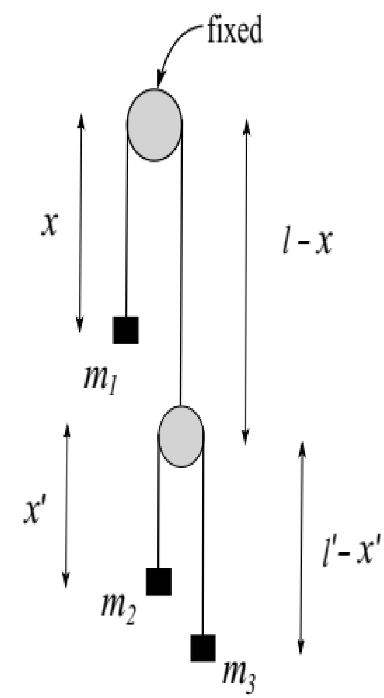

Figure 8: A double Atwood machine. A cable of length l passes over a fixed pulley. Tied to one end of the cable is a mass m_1, while at the other end is a moveable pulley. A cable of length l' passes over this second pulley, with masses m_2, m_3 attached to its two ends. The pulleys and cables are treated as massless, there is no friction, and we assume the radii of the pulleys are negligible compared to l, l'. We may use the coordinates x, x' shown as generalized coordinates.

This system has two degrees of freedom: the vertical displacement x of mass m_1 from the fixed pulley, and the vertical displacement x' of mass m_2 from the moveable pulley. We take these as our generalized coordinates. If we denote the vertical displacements of the three masses from the fixed pulley by x_i, $i = 1, 2, 3$, and similarly the vertical displacement of the moveable pulley from the fixed pulley by x_p, then

$$x_1 = x\ , \qquad x_2 = (l - x) + x'\ , \qquad x_3 = (l - x) + (l' - x')\ , \qquad x_p = l - x\ . \tag{2.78}$$

The kinetic energy is hence

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}' - \dot{x})^2 + \frac{1}{2}m_3(\dot{x}' + \dot{x})^2\ , \tag{2.79}$$

while the potential energy is

$$V = -m_1 g x - m_2 g(l - x + x') - m_3 g(l + l' - x - x')\ . \tag{2.80}$$

In general neither coordinate is ignorable, and Lagrange's equations for x, x' read

$$\begin{aligned} m_1\ddot{x} + m_2(\ddot{x} - \ddot{x}') + m_3(\ddot{x} + \ddot{x}') &= (m_1 - m_2 - m_3)g\ , \\ -m_2(\ddot{x} - \ddot{x}') + m_3(\ddot{x} + \ddot{x}') &= (m_2 - m_3)g\ . \end{aligned} \tag{2.81}$$

A little algebra allows us to solve for the accelerations

$$\begin{aligned} \ddot{x} &= \frac{m_1(m_2 + m_3) - 4m_2m_3}{m_1(m_2 + m_3) + 4m_2m_3}g\ , \\ \ddot{x}' &= \frac{2m_1(m_2 - m_3)}{m_1(m_2 + m_3) + 4m_2m_3}g\ . \end{aligned} \tag{2.82}$$

In particular notice that if $m_2 = m_3$ then $\ddot{x}' = 0$ and x' moves with constant speed. This also follows since in this case x' is an ignorable coordinate, and this statement is conservation of the conjugate momentum $p_{x'}$. Also notice that if $m_1 = m$, $m_2 = m_3 = \frac{m}{2}$ then $\ddot{x} = \ddot{x}' = 0$, and both p_x and $p_{x'}$ are conserved.

Charged particle in an electromagnetic field

Consider a particle of mass m and charge e moving in an electromagnetic field. For our purposes here we don't need to know anything about the theory of electromagnetism, except that the electric field $\mathbf{E}(\mathbf{r}, t)$ and magnetic field $\mathbf{B}(\mathbf{r}, t)$ can be written in terms of a scalar potential $\phi(\mathbf{r}, t)$ and vector potential $\mathbf{A}(\mathbf{r}, t)$ via

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}\ , \qquad \mathbf{B} = \nabla \wedge \mathbf{A}\ . \tag{2.83}$$

The Lagrangian of the charged particle in such a background electromagnetic field is simply

$$L = \frac{1}{2}m|\dot{\mathbf{r}}|^2 - e\left(\phi - \dot{\mathbf{r}} \cdot \mathbf{A}\right)\ . \tag{2.84}$$

The first term is of course the kinetic energy of the particle, while the second term gives rise to the *Lorentz force law*. Let us see this by working out Lagrange's equations. These read

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}}\right) = \frac{\partial L}{\partial \mathbf{r}}\ , \tag{2.85}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{\mathbf{r}} + e\mathbf{A}) = -e\,(\nabla\phi - \nabla(\dot{\mathbf{r}}\cdot\mathbf{A})) \ . \tag{2.86}$$

Recalling the vector calculus identity

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\wedge(\nabla\wedge\mathbf{b}) + \mathbf{b}\wedge(\nabla\wedge\mathbf{a}) \ , \tag{2.87}$$

we may rewrite the last term in (2.86) as

$$\nabla(\dot{\mathbf{r}}\cdot\mathbf{A}) = (\dot{\mathbf{r}}\cdot\nabla)\mathbf{A} + \dot{\mathbf{r}}\wedge(\nabla\wedge\mathbf{A}) \ . \tag{2.88}$$

On the other hand on the left hand side of (2.86), using the chain rule we have the term

$$\dot{\mathbf{A}} = \frac{\partial\mathbf{A}}{\partial t} + (\dot{\mathbf{r}}\cdot\nabla)\mathbf{A} \ . \tag{2.89}$$

The terms involving $(\dot{\mathbf{r}}\cdot\nabla)\mathbf{A}$ cancel in (2.86), which rearranges to give

$$m\ddot{\mathbf{r}} = e\left(-\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}\right) + e\,\dot{\mathbf{r}}\wedge(\nabla\wedge\mathbf{A}) = e\,(\mathbf{E} + \dot{\mathbf{r}}\wedge\mathbf{B}) \ , \tag{2.90}$$

and in the last step we have used (2.83). The force $\mathbf{F} = e(\mathbf{E} + \dot{\mathbf{r}}\wedge\mathbf{B})$ on the right hand side of (2.90) is called the *Lorentz force*. Notice that the corresponding potential term $e(\phi - \dot{\mathbf{r}}\cdot\mathbf{A})$ in the Lagrangian (2.84) depends on the particle's velocity $\dot{\mathbf{r}}$. Note also in this example that the momentum $\mathbf{p}$ conjugate to $\mathbf{r}$ is

$$\mathbf{p} = \frac{\partial L}{\partial\dot{\mathbf{r}}} = m\dot{\mathbf{r}} + e\,\mathbf{A} \ . \tag{2.91}$$

Thus the canonical momentum is the usual linear momentum $m\dot{\mathbf{r}}$, shifted by the charge e times the vector potential $\mathbf{A}$. We thus need to be careful what we mean by "momentum" in this case.

The two-body problem

Let us return to the two-body problem. This is the simplest non-trivial closed system, consisting of two particles with Lagrangian

$$L = \frac{1}{2}m_1|\dot{\mathbf{r}}_1|^2 + \frac{1}{2}m_2|\dot{\mathbf{r}}_2|^2 - V(|\mathbf{r}_1 - \mathbf{r}_2|) \ . \tag{2.92}$$

As explained in section 2.4, being a closed system this enjoys conservation of energy, momentum and angular momentum. Conservation of momentum $\mathbf{P} = m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2$ is equivalent to the centre of mass

$$\mathbf{R} \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \tag{2.93}$$

moving at constant velocity. Thus we may write (choosing $\mathbf{R}(0) = \mathbf{0}$ without loss of generality)

$$\mathbf{R}(t) = \mathbf{V}_0 t \ , \qquad \text{where} \quad \mathbf{V}_0 = \frac{m_1\mathbf{v}_1^{(0)} + m_2\mathbf{v}_2^{(0)}}{m_1 + m_2} \tag{2.94}$$

and $\mathbf{v}_I^{(0)}$ are the initial velocities of the two particles.

The interesting dynamics is described by the separation vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. One computes

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{R} + \frac{m_2}{m_1+m_2}\mathbf{r} , \\ \mathbf{r}_2 &= \mathbf{R} - \frac{m_1}{m_1+m_2}\mathbf{r} . \end{aligned} \qquad (2.95)$$

Notice we have simply changed coordinates from $(\mathbf{r}_1, \mathbf{r}_2) \to (\mathbf{R}, \mathbf{r})$, and the Lagrangian (2.92) becomes

$$L = \frac{1}{2}(m_1+m_2)|\dot{\mathbf{R}}|^2 + \frac{1}{2}\mu|\dot{\mathbf{r}}|^2 - V(|\mathbf{r}|) , \qquad (2.96)$$

where we have defined the *reduced mass* $\mu = m_1 m_2/(m_1+m_2)$. Here $\mathbf{R}$ is ignorable, which leads to conservation of the conjugate momentum $p_{\mathbf{R}} = \mathbf{P}$. We thus effectively reduce the problem to a single particle of mass μ moving in an *external* central potential $V = V(|\mathbf{r}|)$:

$$L_{\text{reduced}} = \frac{1}{2}\mu|\dot{\mathbf{r}}|^2 - V(|\mathbf{r}|) . \qquad (2.97)$$

Since V is rotationally invariant angular momentum is conserved, which follows from conservation of angular momentum for the original two-body problem. Notice we have "lost" translational invariance in the reduced problem (translation acts instead on the coordinate $\mathbf{R}$), and hence also Galilean invariance. The rest of the problem is now familiar from the Dynamics course. Conservation of angular momentum $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$ implies that motion occurs in the fixed plane orthogonal to $\mathbf{L}$. Introducing polar coordinates (ϱ, ϕ) in this plane the Lagrangian (2.97) reads

$$L_{\text{reduced}} = \frac{1}{2}\mu(\dot{\varrho}^2 + \varrho^2\dot{\phi}^2) - V(\varrho) , \qquad (2.98)$$

and since ϕ is ignorable (another consequence of angular momentum conservation) we have that the conjugate momentum $p_\phi = \mu\varrho^2\dot{\phi}$ is conserved. The equation of motion for ϱ is

$$\mu\ddot{\varrho} = -\frac{\partial V}{\partial \varrho} + \mu\varrho\dot{\phi}^2 = -\frac{\partial V}{\partial \varrho} + \frac{p_\phi^2}{\mu\varrho^3} = -\frac{\partial V_{\text{eff}}}{\partial \varrho} , \qquad (2.99)$$

where $V_{\text{eff}}(\varrho) = V(\varrho) + \frac{p_\phi^2}{2\mu\varrho^2}$. Conservation of energy for the Lagrangian $L_{\text{radial}} = \frac{1}{2}\mu\dot{\varrho}^2 - V_{\text{eff}}(\varrho)$ that gives the equation of motion (2.99) is then

$$E = \frac{1}{2}\mu\dot{\varrho}^2 + V_{\text{eff}}(\varrho) = \text{constant} , \qquad (2.100)$$

which leads to a first order ODE for $\varrho(t)$, and we have reduced the problem to quadratures. Notice that it is the large number of conserved quantities (energy, momentum, angular momentum) that have allowed us to reduce the original problem with six degrees of freedom to a single integral equation for $\varrho(t)$. For $V(\varrho) = -\kappa/\varrho$, corresponding to the inverse square law force (1.11), the solutions are conic sections, as shown in the Dynamics course.

2.6 * More on Hamilton's principle

In section 2.2 we introduced a new fundamental principle of classical mechanics: Hamilton's principle of least action. We saw that this is equivalent to Newton's second law for point particles, but given the elegance and universal nature of the principle it is interesting to ask why it should hold. The whole of this subsection is understood as starred.

Historically the principle of least action took some time to develop, and Lagrange's formulation generalized earlier related ideas. However, one intuitive way to think about it is that Nature is lazy. Assuming that energy $E = T + V$ is conserved, then the system is free to transfer energy between kinetic and potential so long as their sum remains constant. Minimizing the integral of $T - V$ then amounts to minimizing the kinetic energy over the path (*i.e.* not moving around too much). At the same time one is then maximizing the potential energy (*i.e.* maximizing the amount of energy that may be converted into other forms).[6] This is a nice picture, but as we already commented the action is in general *not* minimized by the solution to the equations of motion – the latter is just a stationary point.

One can in a sense *derive* the principle of least action from quantum mechanics, although in doing so this really just shifts the question elsewhere. Obviously it's not appropriate to go into a detailed discussion of this here, but the basic ideas (due to Feynman) are simple to state. In the classical principle of least action problem we fixed the endpoints $\mathbf{q}^{(1)} = \mathbf{q}(t_1)$, $\mathbf{q}^{(2)} = \mathbf{q}(t_2)$ at fixed times t_1, t_2, and sought a path $\mathbf{q}(t)$ which extremizes the action S subject to these boundary conditions. This is the *classical path* of the system. In quantum mechanics *all* possible paths between these endpoints play a role. More precisely, each path $\mathbf{q}(t)$ in the configuration space has associated to it the complex phase $\mathrm{e}^{\mathrm{i}S/\hbar}$, where $S = S[\mathbf{q}(t)]$ is the action of the path (2.6) and $\hbar \simeq 1.05 \times 10^{-34}$ Js is (the reduced) *Planck's constant*. Notice that both S and $\hbar$ have units of angular momentum, so it makes sense to take the exponential of the dimensionless quantity $\mathrm{i}S/\hbar$. One then defines the *amplitude* A to get from $(\mathbf{q}^{(1)}, t_1)$ to $(\mathbf{q}^{(2)}, t_2)$ to be the *sum* of these phases over *all* paths. Given the system is observed at $\mathbf{q}^{(1)}$ at time t_1, the *probability* of finding it at $\mathbf{q}^{(2)}$ at time t_2 is given by $|A|^2$.

As you might imagine, defining these things precisely is not so straightforward. In particular we have to "sum" over all paths $\mathbf{q}(t)$ connecting $\mathbf{q}(t_1) = \mathbf{q}^{(1)}$ and $\mathbf{q}(t_2) = \mathbf{q}^{(2)}$. This is an infinite-dimensional space, and such sums are called *path integrals* in quantum mechanics – it's an integral/sum over a space of paths. The basic idea is then that at non-stationary points of S the phases from different paths are very different ($S/\hbar$ is very large for classical systems, as $\hbar$ is so small), and their contributions to the amplitude tend to cancel. On the other hand near to a stationary point of S the phases constructively interfere (since by definition near a stationary point S is constant to first order). Thus the leading contribution to the amplitude A comes from

[6] So the next time someone tells you to get out of bed/off the sofa, tell them that you're simply obeying Hamilton's principle.

the classical path, meaning one is most likely to observe the particle near to where it should be according to the classical equations of motion. Mathematically what we just described is called the *stationary phase approximation* to oscillatory integrals, which is part of the subject of asymptotic analysis.

Hopefully this at least gives a flavour of how the principle of least action fits into quantum mechanics, and why we observe stationary points of S as classical paths. For those that took Part A Quantum Mechanics: one can prove that the above formulation of quantum mechanics (and in particular the probability we defined) is equivalent to the Schrödinger equation and Born interpretation. The interested reader is referred to the book *Quantum Mechanics and Path Integrals*, by R. P. Feynman and A. R. Hibbs.