Basic Mathematics

Lecture 1

Introduction to Set Theory: Definition of Terms

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Introduction to lecture 1

This lecture will define the concept of set theory and introduce basic terms and properties of set theory.

Intended learning outcomes

At the end of this lecture you will be able to;

(i) Explain basic concepts in set theory.

(ii) Carry out operations involving set connectives.

References

These lecture notes should be complemented with relevant topics in (Antony & Robert, 2006; Kahenya, 2017; Spiegel &Robert, 2009; Seymour, 2020).

Introduction

Set theory forms a basis for all branches of mathematics. Set theory is used in all formal descriptions in mathematics. All mathematical concepts are built upon the concept of a set. The concept of set theory is applicable in many areas that requires grouping or partitioning structures e.g. in computer science, business, biology among many other areas. It is a key concept in modern algebra i.e. in areas of group and ring theory.

A set can be referred to as a well-defined collection of objects which are called the members or elements of that set. The criteria for selecting the members of a set must be well defined. Note that elements of a set are enclosed in braces or curly brackets {.....} and it is usual to use an upper-case letter to represent a set.

Examples of sets

a) The set of English vowels $\mathrm{V} = \{\mathrm{a, e, i, o, u}\}$

b) Set of natural numbers; $\boldsymbol{N} = \{1,2,3, \ldots\}$

c) Set of integers; $Z = \{\ldots -3, -2, -1,0,1,2,3, \cdots\}$

d) Set of fruits in a market stall; {mangoes, apples, oranges, tangerines}

Sets are either finite or infinite e.g., set of English vowels is finite. It has five elements. The set of Real numbers is infinite.

The Empty set or Null set is a set with no members at all. It is denoted { } or $\emptyset$.

A set with only one member is called a singleton e.g. set $\mathrm{A} = \{3\}$.

Specifications of Sets

Set can be specified in different ways;

a) List Notation where there is listing of all the members of the set.

Examples:

i) Set A = $\{1,2,3,4,6,8,12,24\}$

ii) Set B = $\{2,4,6,\ldots,30\}$

b) Description or Predicate Notation involves stating a property of the set's elements or members. In general, $\{x|P(x)\}$, where P is some condition or property of the members of the set. Description can be done using the set-builder notation.

Examples:

i) $\{x|x \in Z, x < 19\}$. Read as: 'the set of all x such that x is an integer and is less than 19'

ii) $\{x: x < 11, x \in N\}$. Read as 'the set of all x such that x is less than 11, where x belongs to set of natural numbers.

Terms and Notation used in Set Theory

We have already mentioned certain terms and symbols, or notations used in set theory. Some of them include:

i) {….} – The elements or members of a set are enclosed in braces or curly brackets. Using other brackets may imply something else e.g. (2, 3) may imply a point in the rectangular coordinate system or a 1 by 2 row matrix.

ii) $\{x|x\}$or $\{x: x\}$ read as, '*x is such that x ….*'

iii) $\in$ - read as '*is a member of*' or '*belong to*' e.g. $x \in \boldsymbol{A}$ is read as '*x is a member of set* A'. For example, given set $\boldsymbol{A} = \{2,6,9,12\}$then $2 \in \boldsymbol{A}$

iv) $\notin$- read as '*is not a member of*' or '*does not belong to*' e.g. given set $\boldsymbol{A} = \{2,6,9,12\}$then$7 \notin \boldsymbol{A}$i.e. 7 is not a member of set A.

v) ξ – denote the Universal set. A universal set contains all the elements being discussed. For example, the set of real numbers is a universal set. Sometimes one can use an upper-case letter U to denote the universal set.

vi) $\emptyset$ or { } - denote the empty set or the null set.

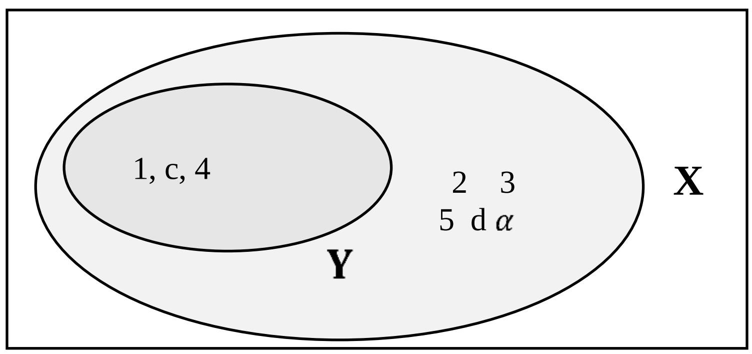
vii) Cardinality refers to the number of elements in a set. The cardinality of a set say set $\boldsymbol{A}$ is written $|\boldsymbol{A}|$ or $\mathrm{n(A)}$. For example given that set **A** = {1,2,r,d,4,5,8,10} then the cardinality of set A is 8 i.e. $|\mathrm{A}| = 8$ or $\mathrm{n(A)} = 8$.

Operations on Sets

Subset

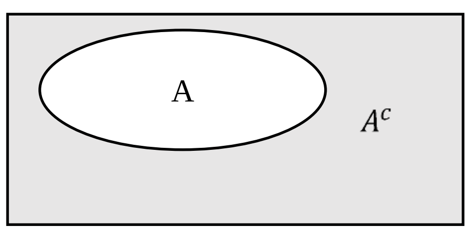
A set say $\boldsymbol{A}$ is said to be a subset of set $\boldsymbol{B}$ if and only if, every element of $\boldsymbol{A}$ is also an element of $\boldsymbol{B}$ denoted $\boldsymbol{A} \subseteq \boldsymbol{B}$. Suppose $\boldsymbol{A} \subseteq \boldsymbol{B}$ and $\boldsymbol{A} \neq \boldsymbol{B}$ then set $\boldsymbol{A}$ is a proper subset of $\boldsymbol{B}$ and written $\boldsymbol{A} \subset \boldsymbol{B}$ i.e. it is strictly less than. Note: the symbol $\subset$ is used when the set is strictly a subset.

Example: Let set $\boldsymbol{X} = \{1,2,3,4,5,c,d,\alpha\}$ and set $\boldsymbol{Y} = \{c,1,4\}$ then $\boldsymbol{Y} \subset \boldsymbol{X}$ i.e. Set Y is contained in set X or Y is a proper subset of set X. The set can be represented in a diagram called a Venn diagram as shown below. Where the rectangle represents the universal set, and its subsets X and Y are represented by ovals (or sometimes circles).



Complement of set

The complement of a set denoted by A' or $\mathrm{A^c}$ or $\neg$ A or $\overline{\mathrm{A}}$ is the set of all elements in the universal set that are not in A. A set A and its complement can be represented in a Venn diagram as shown below.

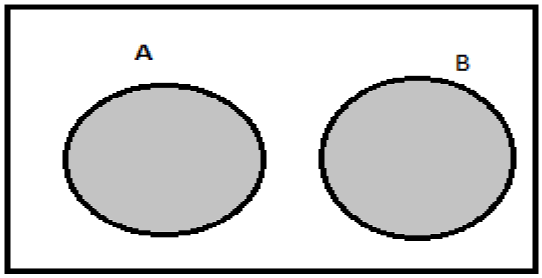


Note that $\mathrm{n(A)} + \mathrm{n(A^c)} = \mathrm{n(\xi)}$, that is, the cardinality of set A together with the cardinality of its complement is equal to the cardinality of the universal set. For example, let the universal set $\xi = \{a,b,c,d,e,f,g,h,j,k,l\}$ and its subset $A = \{a,b,c,d,e,f\}$ then the complement of set A is $A^C = \{g,h,j,k,l\}$. The cardinality of set A, $|A| = 6$. The cardinality of its complement is $|\mathrm{A^c}| =$

5. The universal set has 11 elements hence its cardinality is 11. Again, if the universal set $\xi = \{1,2,3,4,5,6,7,8,9\}$ and its subset $M = \{1,2,3,4,5,6\}$then $M' = \{7,8,9\}$.

Disjoint sets

Two sets A and B are said to be disjoint if they have no common element. For example, if we have set A = {even numbers} and set B = {odd numbers}, then sets A and B are disjoint. There is no number that is both even and odd. Sets A and B in the Venn diagram below are disjoint sets.



Intersection of sets

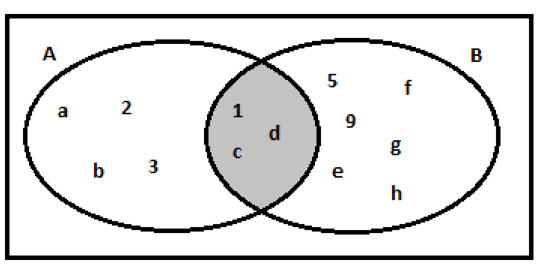
Two sets A and B, which are not disjoint, must have at least one element in common. The set of common elements is the intersection of sets A and B. The intersection of A and B is written,

$A \cap B$ and read as 'A intersection B' or 'A and B'. Note that if two sets are disjoint then their intersection is an empty set.

Example 1: Let set $A = \{13,19,63, a, b, c, d\}$ and $B = \{10,20,30,4,5\}$ then $A \cap B = \emptyset$.

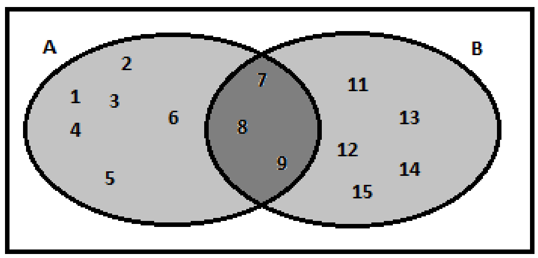
Example 2: Let set $A = \{a, b, c, d, 1,2\ 3\}$ and $B = \{1,5,9, c, d, e, f, g, h\}$ then $A \cap B = \{1, c, d\}$.

The shaded part in the Venn diagram below represents the intersection of sets A and B.



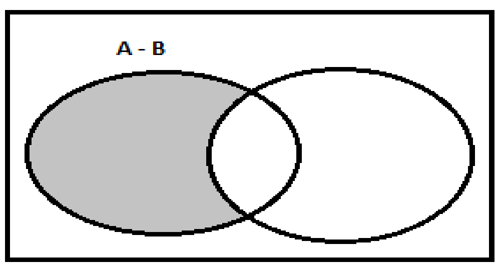
Union of sets

The union of two sets A and B is the set of elements that belong either to A or B or to both A and B. The union of sets A and B is written, $A \cup B$. Read as, 'A union B'. For example, If $A = \{1,2,3,4,5,6,7,8,9\}$ and $B = \{7,8,9,11,12,15\}$ then $A \cup B = \{1,2,3,4,5,6,7,8,9,11,12,15,16,17\}$. Note that their intersection is $A \cap B = \{7,8,9\}$. The shaded region in the Venn diagram below represents the union of sets A and B.



Relative Complement

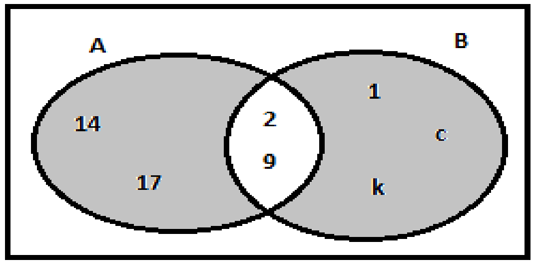
The relative complement of a set say set A and set B denoted $A - B$ or A/B is the elements that belongs to set A and not to set B. The shaded region represents the set A – B.



For example, let set $A = \{1,2,3,4,5,6,7\}$ and set $B = \{1,2,5,8\}$ then $(A - B) = \{3,4,6,7\}$. On the other hand, B – A = {8}. The set A – B can be represented in a Venn diagram as shown below.

Symmetric difference

Suppose we have sets **A** and B**,** then the symmetric difference denoted $A \Delta B$ are elements that belongs to set **A** or **B** but not in their intersection. For example, Let set $A = \{2,9,14,17\}$ and $B = \{1,9,2,c,k\}$ then $A \Delta B = \{1,14,17,c,k\}$ (see the Venn diagram below).



Equivalent sets

If set $A = \{g, w, 9\}$then the cardinality of A denoted $n(A)$ or $|A|$is the number of elements of set A i.e. $n(A) = 3$. Two sets are said to be equivalent if they have the same cardinality. For example, If $A = \{1,2,3,4,9\}$and $B = (a, b, c, w, z\}$then A and B are equivalent since $n(A) = n(B) = 5$, We write $A \equiv B$to mean A is equivalent to B.

Identical sets

Two sets are said to be identical if and only if they have the same members, that is, for any two sets A and B then $A = B$ iff $\forall x, x \in A \Leftrightarrow x \in B$.

Power set

The set of all subsets of a set A is called the power set of A and denoted as $\wp(A)$ or 2^A (NB: $\wp$- Weierstrass elliptic symbol).

For example, If $A = \{a, b\}$ then $\wp(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Note that the empty set is a subset of all sets.

Some basic properties of sets

For some finite sets(non-empty sets) A, B and C.

i) Idempotent properties i.e. $A \cap A = A$ and $A \cup A = A$.

ii) Identity property i.e. $A \cap \emptyset = \emptyset$ and $A \cup \emptyset = A$.

iii) Involution property $(A^c)^c = A$ (or double complement property).

iv) Complements properties i.e.

- $A \cap A^c = \emptyset$ and $A \cup A^c = \xi$ (universal set).
- $\emptyset^c = \xi$ – the complement of the empty set is the unversal set.
- $\xi^c = \emptyset$ – the complement of the universal set is the empty set.

v) Intersection and union of sets is commutative i.e. $A \cap B = B \cap A$ and $A \cup B = B \cup A$.

vi) Intersection and union of sets is associative i.e. $(A \cap B) \cap C = A \cap (B \cap C)$ and $(A \cup B) \cup C = A \cup (B \cup C)$.

vii) Intersection is distributive over union i.e. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

viii) Union is distributive over intersection i.e. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

ix) De Morgan's law i.e. The union and Intersection of sets, interchange under complementation. For example for any sets A and B then $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.

x) Principle of inclusion and exclusion i.e. $|A \cup B| = |A| + |B| - |A \cap B|$ or $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| + |B \cap C| + |A \cap B \cap C|$

Exercise

1) Define the following terms as used in set theory; set, universal set, union of sets, intersection of sets, empty set, cardinality of a set, a subset of a set, absolute complement of a set, relative complement of a set.

2) Determine the power set of set A given that set $A = \{1,2,3, d\}$.

3) Represent the set $A \cap (B \cup C)$ in a Venn diagram by shading the appropriate region.

4) Let the universal set $U = \{0,1,2,3,4,7,8,10,13,12,15,16,17,18,24,23,20,14, 31\}$ and its subsets A,B, and C are such that $A = \{\text{Even numbers}\}$, B = {Prime Numbers} and C = {Divisors of 36}. List the members of the following sets;

(i)	A,B,C	(v)	$(A \cup B) \cap C^c$
(ii)	$B \cap C^c$	(vi)	$A \cup C$
(iii)	$A \cap B \cap C$	(vii)	$A \cap C$
(iv)	$\neg C$	(viii)	$B \cap C$

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