Basic Mathematics

Lecture 3

Definition of the Real Number System and its Properties

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Introduction to Lecture 3

This lecture will introduce to you the Real Number system and its properties. It will define the Real set and the axioms associated with the real number system. The lecture will also cover the absolute value of the real number and demonstrate application of these properties.

Intended learning outcomes

At the end of this lecture you will be able to;

(i) Define the Real number system.

(ii) Outline the properties of the Real number system.

(iii) Define the absolute value of real number.

(iv) Solve problems involving the properties of the real number system.

References

These lecture notes should be complemented with relevant topics in (Antony & Robert, 2006; Kahenya, 2017; Spiegel & Robert, 2009; Seymour, 2020)

Set of Real Numbers

In our previous lectures 1 and 2 we discussed the concept of set. One application of set theory is in understanding the different categories of numbers.

In this lecture we look at what we call the set of Real numbers denoted $\mathbb{R}$, its subsets, and its properties. The subsets of the set of Real numbers are;

a) Natural /counting/whole numbers $\mathbb{N}$, are numbers that we normally use for counting i.e. $\mathbb{N} = \{1,2,3,\ldots\}$. This is the smallest set of Real numbers i.e. it is the set with least elements or numbers. Note that some authors include zero in the set of Natural numbers.

b) Integers $\mathbb{Z}$ are the positive and negative whole numbers and zero i.e.

$$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1,2,3, \cdots\} = \{0, \pm 1,2,3, \cdots\}$$

Integers can also be written as fractions e.g. $2 = \frac{2}{1}$. The set of Naturals numbers is a subset of integers i.e. $\mathbb{N} \subset \mathbb{Z}$.

c) Rational numbers $\mathbb{Q}$ are numbers of the form; $\frac{p}{q}$

where p and q are integers with no common factor and $q \neq 0$ e.g. ¼, ⅔. Note that;

i) $\frac{0}{p} = 0$ where $p \neq 0$

ii) $\frac{p}{0}$ is undefined, where $p \neq 0$

iii) $\frac{0}{0}$ is indeterminate i.e. it has different meanings in different context.

Again note that the set of integers is a subset of rational numbers i.e. $\mathbb{Z} \subset \mathbb{Q}$

d) Irrational numbers $\mathbb{Q}^c$ are numbers that cannot be written as fractions. Irrational numbers are numbers whose decimals are neither terminating nor recurring e.g. $\sqrt{2}$ or roots of prime numbers, π among others.

Thus the set of Real numbers $\mathbb{R}$ is the set that contains ALL the irrational numbers $\mathbb{Q}^c$, rational numbers $\mathbb{Q}$, integers $\mathbb{Z}$, and natural numbers $\mathbb{N}$.

e) Complex numbers $\mathbb{C}$ - A complex number z is an ordered pair (x, y) of real numbers, such that $z = (x, y)$ where $x = Re\, z$ and $y = Im\, z$ e.g. $2 + 5i$.

Note that $\sqrt{-1} = i$ where i is the imaginary number. This implies that $i^2 = -1$.

The set of complex numbers is therefore the largest set of numbers.

Properties of Real Numbers System

The set of the Real numbers and the operations of addition (+) and multiplication $(\cdot)$ are referred to as the Real number system i.e. $(\mathbb{R}, +, \cdot)$. The following are the properties or axioms of real number system:

a) **Closure property**

The set of real numbers is closed under addition and multiplication operations i.e.

$\forall x, y \in \mathbb{R},\ (x + y) \in \mathbb{R}$(Closed under addition).

Again $xy \in \mathbb{R}, \forall x, y \in \mathbb{R}$ (closed under multiplication).

For example given two real numbers say $3 \in \mathbb{R}$ and $8 \in \mathbb{R}$ then $(3 + 8) = 11 \in \mathbb{R}$.

b) **Commutative property**

Real numbers commutate under multiplication and addition i.e. $(x + y) = (y + x), \forall x, y \in \mathbb{R}$ and $xy = yx, \forall x, y \in \mathbb{R}$. For example 3 + 6 = 6 + 3.

c) **Associative property**

Real numbers are associative under the operations of addition and multiplication i.e.

$(x + y) + z = x + (y + z), \forall x, y, z \in \mathbb{R}$and $(xy)z = x(yz), \forall x, y, z \in \mathbb{R}$. For example;

(3+4)+5 = 3 +(4+5)

7+5 = 3+ 9

12= 12

d) **Distributive property**

Real numbers obey the left and distributive property i.e. $x(y + z) = xy + xz, \forall x, y, z \in \mathbb{R}$ - left distributive and $(x + y)z = xz + yz, \forall x, y, z \in \mathbb{R}$ - Right distributive.

e) **Identity property**

There exist two identities in the real number system i.e., the additive and multiplicative identity.

Additive identity

There exist a unique real number 0 (zero) such that $x + 0 = x$ and $0 + x = x \forall x \in \mathbb{R}$, 0 is called the additive identity.

For example; 5 + 0 = 0 + 5 = 5

Multiplicative identity

There exist a unique real number 1 such that $x \cdot 1 = 1 \cdot x = x \forall x \in \mathbb{R}$, 1 is called the multiplicative identity. For example; $9 \cdot 1 = 1 \cdot 9 = 9$

f) **Inverse property**

Additive inverse

There exist a unique real number $(-x)$ such that ; $x + (-x) = (-x) + x = 0 \ \forall x \in \mathbb{R}$.

$(-x)$ is called that additive inverse.

For example; $7 + (-7) = (-7) + 7 = 0$.

Multiplicative inverse

If $x \neq 0$, $\forall x \in \mathbb{R}$ there exist a unique real number; $\frac{1}{x}$ or x^{-1} such that; $x \cdot \frac{1}{x} = \frac{1}{x} \cdot x = 1$. $\frac{1}{x}$ or x^{-1} is called the multiplicative inverse.

For example the multiplicative inverse of 8 is $\frac{1}{8}$ or 8^{-1} such that; $8 \cdot \frac{1}{8} = \frac{1}{8} \cdot 8 = 1$.

Absolute Value of a Real Number

Absolute value of a real number refers to its magnitude. That is, the distance between the origin and the point on the real number line.

The absolute value of the real number x is denoted $|x|$ i.e.,

For all $x \in \mathbb{R}$ then the absolute value of x is given as;

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

Properties of absolute value for any Real numbers x and y

a) $|x| \geq 0, \forall x \in \mathbb{R}$

b) $|x||y| = |xy| \ \forall x, y \in \mathbb{R}$

c) $|-x| = |x|, \forall x \in \mathbb{R}$

d) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}, y \neq 0 \ \forall x, y \in \mathbb{R}$

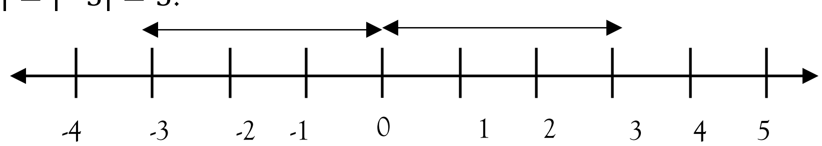
e) $|x + y| \leq |x| + |y|$, $\forall x, y \in \mathbb{R}$ - Triangle inequality.

Application

For example the distance of 3 from the origin written as $|3|$ is 3 i.e. the absolute value of 3 is 3.

Again the distance of – 3 from the origin is written as $|-3|$ is 3 i.e. absolute value of – 3 is 3.

Hence $|3| = |-3| = 3$.



In general, given a real number x and $|x| = b$ then $x = b$ or $x = -b$

Example 1: Determine the value of x given that $|x| = -9$

Solution: This is invalid since the absolute value cannot be a negative number.

Example 2: Find the value of x if $|x| = 11$.

Solution: $x = 11$ or $x = -11$. The two real numbers are same distance from the origin on the real number line.

Suppose that $|x| < 9$. This means that the real number x must be less than 9 units from the origin. This implies that one need to find all real numbers whose distance from the origin is always less than 9. Hence it means that $x < 9$ or $x > -9$ i.e. x must be in the interval $-9 < x < 9$.

In general, given a real number x and $|x| < b$ then $x < b$ or $x > -b$

Example 2: Determine the integers x that satisfy the following; $|x| \leq 5$.

Solution: by definition, $-5 \leq x \leq 5 \Rightarrow x = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$

Suppose that $|x| > 4$. This means that the real number x must be greater than 4 units from the origin. This means one need to find all real numbers whose distance from the origin is always greater than 4 i.e. $x > 4$ or $x < -4$,

In general, given a real number x and $|x| > b$ then $x > b$ or $x < -b$

Example 3: Determine integers x that satisfy the following $|x| > 11$

Solution: Using the definition we have; $|x| > 11 \Rightarrow x > 11$ or $x < -11$.

That is; $x = \{12, 13, \cdots\} \cup \{\cdots - 13, -12\} = \{\pm 12, 13, 14, \cdots\}$.

Example 4:

a) Solve; $|x - 7| = 5$

Solution: Let $a = x - 7 \Rightarrow |a| = 5 \Rightarrow a = 5$ or $a = -5$. Hence $x - 7 = 5 \therefore x = 12$ or $x - 7 = -5 \therefore x = 2$

b) Solve; $|2x - 3| < 11$

Solution: Let $a = 2x - 3 \Rightarrow |a| < 11$

Hence $a < 11$ or $a > -11$ thus $2x - 3 < 11 \Rightarrow 2x < 14 \therefore x < 7$ or $2x - 3 > -11 \Rightarrow 2x > -8 \therefore x > -4$.

Therefore we have $-4 < x < 7$.

c) Solve; $|x + 5| > 8$

Solution: Let $a = x + 5 \Rightarrow |a| > 8 \Rightarrow a > 8$ or $a < -8$

Hence $x + 5 > 8 \Rightarrow x > 3$ or $x + 5 < -8 \Rightarrow x < -13$.

d) Solve for x in the following; $4|x - 3| = 48$

Solution: Divide every side by 4 to get;

$$|x - 3| = 12$$

either $x - 3 = 12 \Rightarrow x = 15$

or $x - 3 = -12 \Rightarrow x = -9$

$$\therefore x = \{-9, 15\}$$

e) Solve; $|x - 11| = |x + 17|$

Solution: Either $x - 11 = x + 17 \cdots$ (i) or $x - 11 = -(x + 17) \cdots$ (ii)

$$\text{For equation (i) we have: } x - x = 11 + 17$$

$$0 = 27 - \text{invalid}$$

$$\text{For equation (ii)we have: } x - 11 = -x - 17$$

$$2x = 11 - 17 = -6$$

$$2x = -6 \therefore x = -3$$

i.e. the inequality is only valid when x = - 3

f) Solve: $|x - 5| < 13$

Solution: Either;

(i) $x - 5 < 13 \Rightarrow x < 18$ or

(ii) $x - 5 > -13 \Rightarrow x > -8$

Therefore, we have; $-8 < x < 18$

g) Solve; $|x + 2| \geq 17$ (2 marks)

Either

(i) $x + 2 \geq 17 \Rightarrow x \geq 15$ or

(ii) $x + 2 \leq -17 \Rightarrow x \leq -19$

Exercise

1. Solve the following;

a) $|x| = -23$ b) $\frac{|x+4|}{3} = 4$ c) $\left|\frac{3x}{2}\right| = 12$

d) $\left|\frac{1-x}{-3}\right| = 2|x-3|$ e) $\frac{|x-3|}{-5} = |3-7x|$ f) $|5x+1| = 11$

2. Solve the following;

a) $|x-2| < 12$

b) $|5x+5| < 11$

c) $|4x-5| < 13$

d) $\frac{1}{2}|5x-7| > 3$

e) $\frac{2}{3}|3x+8| > 10$

f) $3|x-4| \geq 18$

3. Prove that $|x+y| \leq |x| + |y|$

Bibliography

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