Basic Mathematics

Lecture 4

Solving linear systems: Elimination and substitution methods

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Introduction to Lecture 4

This lecture will focus on how to solve linear systems with 2 or 3 unknown using the methods of elimination, substitution, graphical method (for a system with two unknowns).

Intended learning outcomes

At the end of this lecture you will be able to;

(i) Solve linear systems using the elimination and substitution methods.
(ii) Apply the methods to solve real-life problems.

References

These lecture notes should be complemented with relevant topics in (Antony & Robert, 2006; Kahenya, 2017; Spiegel & Robert, 2009)

Introduction

Linear equations are equations whose function if plotted on a plane you get a straight line. In the 3D you get a plane. A linear equation is a first-degree polynomial. It may have one or more unknowns or variables.

Some quantities in real-life may be well understood if expressed in linear form but it is rare to have linear relationships because of the many parameters that are in play. Linear equations are used in real life to explain or understand situations while holding some variables constants. For example, the price p of a commodity may partly depend on demand q and partly constant. This may be written as $p = bq + k$.

A linear equation with two unknowns or variables x and y is of the form; $\mathrm{ax + by = c}$ where a, b, and c are known constants with $a \neq 0, b \neq 0$. For example; $3x + 4y = 7$.

A linear equation with three unknowns or variables x, y, and z is of the form; $\mathrm{ax + by + cz = d}$ where a, b, c, and d are known constants with $a \neq 0, b \neq 0, c \neq 0$.

For example; $3x + 4y + 5z = 9$.

One can have a system of linear equations with 2 or 3 variables that can be solved simultaneously. The system may have a unique solution (exactly one solution), infinitely many solutions, or no solutions (an inconsistent system).

Elimination method

In this method the goal is to eliminate one of the unknown to work with only one unknown.

Example 1: Solve the following system using purely elimination method; $$\begin{aligned} 3x + 5y &= 21 \cdots \text{(i)} \\ 5x - 2y &= 4 \cdots \text{(ii)} \end{aligned}$$

Solution: Suppose we eliminate x first. The coefficient of x in equation (i) is 2 and in equation (ii) is 5. We therefore need to multiply equation (i) with 4 and equation (ii) with 3 to ensure that the coefficients of x in both equations are the same. Then we subtract i.e.

$$(3x + 5y = 21) \times 5$$
$$(5x - 2y = 4) \times 3$$

We get;

$$\begin{array}{l} 15x + 25y = 105 \\ \underline{15x - 6y = 12\ -}\ \therefore y = 3 \\ \quad 31y = 93 \end{array}$$

Since we are dealing with pure elimination method; we need to multiply equation (i) with 2 and equation (ii) 5 to get (we aim to ensure that the coefficients of y are the same).

$$(3x + 5y = 21) \times 2$$
$$(5x - 2y = 4) \times 5$$

To get; $$\begin{array}{l} 6x + 10y = 42 \\ \underline{25x - 10y = 20\ +}\ \therefore x = 2 \\ \quad 31x = 62 \end{array}$$

Example 2: Solve the following using elimination method; $$\begin{aligned} 7x - y &= 10 \\ 5x + 3y &= 22 \end{aligned}$$

To eliminate x, multiply 1st equation by 5 and 2nd equation by 7 and **subtract** 1st equation from the 2nd one i.e.

$$\begin{array}{c} 35x - 5y = 50 \\ \underline{35x + 21y = 154} \\ -26y = -104 \end{array}$$

Dividing both sides by -26 to get y = 4

To eliminate y, multiply 1st equation by 3 and **add** the two equations i.e.

$$\begin{array}{c} 21x - 3y = 30 \\ \underline{5x + 3y = 22} \\ 26x = 52 \end{array}$$

$$\therefore x = 2$$

Example 3: Solve the following system using elimination method;

$$x + y - z = 1 \cdots (i)$$
$$x - 2y - 3z = -16 \cdots (ii)$$
$$x + 3y + z = 15 \cdots (iii)$$

Solution: Subtract (i) from (ii) to get;

$$\begin{array}{c} x + 3y + z = 15 \\ \underline{x + y - z = 1} \\ 2y + 2z = 14 \cdots (iv) \end{array}$$

We need to get another equation with two variables y and z. thus we can subtract (i) from (ii) to get;

$$\begin{array}{c} x - 2y - 3z = -16 \\ \underline{x + y - z = 1} \\ -3y - 2z = -17 \cdots (v) \end{array}$$

Next we add equations (iv) and (v) to eliminate z i.e.

$$\begin{array}{c} 2y + 2z = 14 \\ \underline{-3y - 2z = -17} \\ -y = -3 \\ \therefore y = 3 \end{array}$$

from equation (iv) $2(3) + 2z = 14 => 2z = 14 - 6 = 8 \therefore z = 4$

From equation (i) we get; $x + 3 - 4 = 1 \therefore x = 2$

Substitution method

Substitution can also be referred to as replacing method. The goal in this method is to replace one variable in the equation such that the equation to have only one unknown (i.e. in an equation with two unknowns).

Example 1: Solve using substitution method:

$$x + 3y = 7 \cdots (i)$$
$$3x - 2y = -1 \cdots (ii)$$

Solution: From equation (i) above we can make x the subject i.e. $x = 7 - 3y$

Next we replace/substitute x with (7 – 3y) in equation (ii) to get;

$$3(7 - 3y) - 2y = -1 => 21 - 9y - 2y = -1$$

$$21 - 11y = -1 => -11y = -22 \therefore y = 2$$

Next we replace y with 2 in equation (i) to get; $x + 3(2) = 7 => x + 6 = 7 \therefore x = -1$

Our solution is unique i.e. $\{x, y\} = \{-1, 2\}$

Example 2: Solve the following linear system using substitution method; $\begin{array}{r} x + 2y = 15 \cdots \text{(i)} \\ 2x + 4y = 40 \cdots \text{(ii)} \end{array}$

Solution:

From equation (i) $x = 15 - 2y$.

Next we replace x equation (ii) with (15 - 2y) to get; $2(15 - 2y) + 4y = 40$

$$30 - 4y + 4y = 40$$

$$30 = 40 - \text{invalid}$$

This implies that the system has no solutions.

The two equations will form two parallel lines when plotted on a plane.

Example 3: Solve the following linear system using substitution method; $\begin{array}{r} x + 2y = 15 \cdots \text{(i)} \\ 2x + 4y = 30 \cdots \text{(ii)} \end{array}$

Solution:

From equation (i) $x = 15 - 2y$.

Next we replace x equation (ii) with (15 - 2y) to get; $2(15 - 2y) + 4y = 30$

$$30 - 4y + 4y = 40$$

$$30 = 30 - \text{valid}$$

This implies that the system has many solutions. A keen introspection of the system will note that equation (ii) is a multiple of equation (i). The two equations will form a coincident line when plotted on a plane.

Example 4: Anita bought 7 mangoes and 4 pears at a total cost of 130/-. Chris bought 11 similar mangoes and 5 similar pears at a total cost 185/-. Find how much Kemboi paid for 12 similar mangoes and 8 similar pears.

$$\begin{array}{c} 7m + 4p = 130 \\ 11m + 5p = 185 \end{array}$$

From the first equation we get;

$$\begin{array}{c} 7m = 130 - 4p \\ m = \dfrac{130 - 4p}{7} \end{array}$$

Substitute in the second equation to get;

$$11\left(\frac{130 - 4p}{7}\right) + 5p = 185$$

$$\frac{1430-44p}{7}+5p=185$$

Multiply every term by 7 to get;

$$1430-44p+35p=1295$$
$$-9p=-135$$
$$\therefore p=15/- \text{ per pear}$$
$$\text{hence; } m=\frac{130-4p}{7}=\frac{130-60}{7}=\frac{70}{7}=10/- \text{ per mango.}$$

Therefore Kemboi paid;

$$12\times 10+8\times 15=120+120=240/-$$

Exercise

1) Onyango bought 3 mangoes and 2 oranges for 84 shillings. Achieng bought 5 similar and 3 similar oranges for 130 shillings. Determine the cost of a mango and an orange.

2) Machines A and B takes 3 hrs to complete a task. Machines A and C takes 4 hrs to complete the same task. While machines B and C takes 6 hrs to complete the same task. Determine how long each would complete the task when working alone.

3) Solve the following using the substitution method;

a) $\begin{matrix}2x-y=2\\ x+2y=11\end{matrix}$ b) $\begin{matrix}5x+2y=9\\ 6x+y=8\end{matrix}$ c) $\begin{matrix}2x+y+z=7\\ x-y+2z=5\\ x+3y-z=4\end{matrix}$

4) Solve the following systems using elimination method:

a) $\begin{matrix}7x+5y=15\\ x-y=9\end{matrix}$ b) $\begin{matrix}9x-2y=19\\ 3x+2y=17\end{matrix}$ c) $\begin{matrix}x+y+z=5\\ 3x+y-z=-7\\ x-y-4z=-21\end{matrix}$

5) Solve the following systems:

a) $\begin{matrix}x+y=3\\ 3x-y=-11\end{matrix}$ b) $\begin{matrix}3x+y=7\\ 4x+3y=6\end{matrix}$ c) $\begin{matrix}2x+3y=14\\ y-3x=1\end{matrix}$

Bibliography

Antony, C., & Robert, D. (2006). *Foundation Maths*. Prentice Hall.

Kahenya, P. (2017). *Foundation Maths*. LAP Lambert Academic Publishers.

Murray, S., & Robert, M. (2009). *College Algebra*. McGraw-Hill.