Basic Mathematics

Lecture 6

Solving quadratic equations: Factorization method

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Introduction to lecture 6

This lecture discusses how to solve quadratic equations using the factorization method.

Intended learning outcomes

At the end of this lecture you will be able to;

(i) Define a quadratic equation.
(ii) Solve quadratic equations using factorization method.

References

These lecture notes should be complemented with relevant topics in (Antony & Robert, 2006; Kahenya, 2017; Spiegel & Robert, 2009)

Introduction

A polynomial in x of degree n is of the form; $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n = k$ where $a_0, a_1, a_2, \cdots a_n$ and k are constant, variable x.

A polynomial of degree 2 is called a quadratic equation and it is of the form;

$$a_0 + a_1x + a_2x^2 = k \text{ where } a_n \neq 0$$

In this lecture we shall use the expression $ax^2 + bx + c = 0$ as the general quadratic equation with variable x. For example; $3x^2 + 5x + 3 = 0; 7x^2 + 3 = 0; 9x^2 - 7x = 0$.

The value of x will give the solution to the quadratic equation. The solution is also called the root of the equation.

A quadratic equation may be solved by factorization method, quadratic formula, completing the square method or graphical method.

Factorization Method

Factorization method involve simplifying a quadratic equation and expressing it as a product of binomials i.e.

$$ax^2 + bx + c = (x + k_1)(x + k_2) = 0, \text{where } k_1 \text{and } k_2 \text{ are known constants}$$

The procedure of factorizing $ax^2 + bx + c$ involving finding two terms $T_1 and\ T_2$ such that;

(i) Their sum is b i.e., $T_1 + T_2 = b$ and
(ii) Their product is ac i.e., $T_1 \times T_2 = ac$

Then replace the middle term bx with $T_1x + T_2x$ to get $ax^2 + T_1x + T_2x + c$

Then apply group factorization on $ax^2 + T_1x + T_2x + c$.

Note

(i) After identifying the terms $T_1$and T_2 we can proceed directly to get the roots of $ax^2 + bx + c = 0$. The roots of the equation will be as;
- If $a = 1$ then the roots $\{x_1, x_2\} = \{-T_1, -T_2\}$.
- If $a \neq 1$ then the roots $\{x_1, x_2\} = \left\{-\frac{T_1}{a}, -\frac{T_2}{a}\right\}$.

(ii) Not all quadratic equations can be solved by factorization method. It is a weak method.

Example 1: Factorize the quadratic expression: $x^2 + 5x + 6$

Solution: This is= a quadratic expression of the form $ax^2 + bx + c$. Our a = 1, b = 5, and c = 6.

We find two terms $T_1$and T_2 such that;

$\text{Sum} = (T_1 + T_2) = b = 5$ and $\text{Product} = (T_1T_2) = ac = (1 \times 6) = 6$.

We manipulate the factors of 6 i.e., 1, 2,3,6 to see what combination can give us a sum of 5 and product of 6.

Clearly $T_1 = 2, T_2 = 3$ since $(T_1 + T_2) = 2 + 3 = 5$ and $T_1T_2 = 1 \times 6 = 6.$

Hence, we have;

$$x^2 + 5x + 6 = x^2 + 2x + 3x + 6$$

Next perform group factorization on the RHS to get;

$$= x(x + 2) + 3(x + 2)$$

$$= (x + 2)(x + 3)$$

Example 2: Factorize the expression: $4x^2 + 15x + 9$

Solution: We find two terms such that;

$\text{Sum} = T_1 + T_2 = 15$ and $\text{Product} = T_1T_2 = 4 \times 9 = 36$ clearly $T_1 = 12, T_2 = 3$

Hence, we have; $4x^2 + 15x + 9 = 4x^2 + 12x + 3x + 9$

We next perform group factorization on the RHS to get

$$= 4x(x + 3) + 3(x + 3)$$

$$= (x + 3)(4x + 3)$$

Example 3: Solve the equation by factorization method: $x^2 + 4x - 21 = 0$

Solution: We find two terms such that;

$\text{Sum} = T_1 + T_2 = 4$ and $\text{Product} = T_1 T_2 = -21$ clearly $T_1 = 7, T_2 = -3$

Hence we have; $x^2 + 4x - 21 = x^2 + 7x - 3x - 21 = 0$

We next perform group factorization on the RHS to get;

$$= x(x + 7) - 3(x + 7) = 0$$

$$= (x + 7)(x - 3) = 0$$

Either $(x + 7) = 0$ or $(x - 3) = 0 \Rightarrow x = -7$ or $x = 3$

Alternatively;

We find two terms such that;

$\text{Sum} = T_1 + T_2 = 4$ and $\text{Product} = T_1 T_2 = -21$ clearly $T_1 = 7, T_2 = -3$

Since our $a = 1$ then the roots are $x = -7$ or $x = 3$

Example 4: Solve the equation; $3x^2 + x - 24 = 0$

Solution: We find two terms such that;

$\text{Sum} = T_1 + T_2 = 1$ and $\text{Product} = T_1 T_2 = -72$ clearly $T_1 = 9, T_2 = -8$

Hence, we have;

$$3x^2 + x - 24 = 3x^2 + 9x - 8x - 24 = 0$$

We next perform group factorization on the RHS to get

$$= 3x(x + 3) - 8(x + 3) = 0$$

$$= (x + 3)(3x - 8) = 0$$

Either $(x + 3) = 0$ or $(3x - 8) = 0 \Rightarrow x = -3$ or $x = \frac{8}{3}$

Alternatively;

We find two terms such that;

$\text{Sum} = T_1 + T_2 = 1$ and $\text{Product} = T_1 T_2 = -72$ clearly $T_1 = 9, T_2 = -8$

Since our $a = 3 \neq 1$ then the roots are $x = -\frac{9}{3} = -3$ or $x = \frac{8}{3}$

Example 5: Given the roots $x = -11$ and $x = 7$ determine corresponding equations of the form $ax^2 + bx + c = 0$

Solution: Given $x = -11 => (x + 11) = 0$ and $x = 7 => (x - 7) = 0$

Hence, we have; $(x + 11)(x - 7) = 0$

$$x(x - 7) + 11(x - 7) = 0$$

$$x^2 - 7x + 11x - 77 = 0$$

$$x^2 + 4x - 77 = 0$$

Alternatively

Given the roots α and β then $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

In our case above we let $\alpha = -11, \beta = 7$

Then $x^2 - (\alpha + \beta)x + \alpha\beta = x^2 - (-11 + 7)x + (-11)(7) = 0$

$$x^2 + 4x - 77 = 0$$

In general, if α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$ then;

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

Example 6: The square of a number is 28 more than thrice the number. Find the number(s).

Solution: Let the number be x. Then we have; $x^2 = 3x + 28 -$ a quadratic equation

$$x^2 - 3x - 28 = 0$$

We have $T_1 + T_2 = -3,$ and $T_1 T_2 = -28 => T_1 = 4, T_2 = -7$

Hence,

$$x^2 - 3x - 28 = x^2 + 4x - 7x - 28 = 0$$

$$= x(x + 4) - 7(x + 4) = 0$$

$$= (x + 4)(x - 7) = 0$$

$$x + 4 = 0 => x = -4 \text{ or } x - 7 = 0 => x = 7$$

Example 7: Determine the corresponding quadratic equation of the form $ax^2 + bx + c = 0$ given the roots; $x = 2i$ and $x = 2 - 3i$.

Solution: From what is given we can rewrite it as; $(x - 2i) = 0$ or $x - (2 - 3i) = 0$

Hence we have; $[x - 2i][x - (2 - 3i)] = 0$

The product of the two is zero i.e., $[x - 2i][x - 2 + 3i] = 0$

$$x(x - 2 + 3i) - 2i(x - 2 + 3i) = 0$$
$$x^2 - 2x - 3xi - 2xi + 4i - 6i^2 = 0$$
$$x^2 - (2 + 5i)x + (4i + 6) = 0$$

Note that; $i^2 = -1$. where i is the imaginary number (of the complex set)

Example 8: Determine the corresponding quadratic equation of the form $ax^2 + bx + c = 0$ given the roots; $x = \frac{1}{2}, x = \frac{3}{7}$

Solution: If α and β are two roots, then the corresponging quadratic equation is of the form;

$$x^2 - (\alpha + \beta)x + (\alpha\beta) = 0$$

In this case; $\alpha = \frac{1}{2}$ and $\beta = \frac{3}{7}$. Hence substituting accordingly we have;

$$x^2 - \left(\frac{1}{2} + \frac{3}{7}\right)x + \left(\frac{1}{2} \times \frac{3}{7}\right) = 0$$

$$x^2 - \frac{13}{14}x + \frac{3}{14} = 0$$

You can also write it as; $14x^2 - 13x + 3 = 0$

Example 9: Given that α and β are the roots of the equation $x^2 + 7x + 4 = 0$, find the values of;

(i) $(\alpha + \beta)$ and $\alpha\beta$

(ii) $(\alpha - \beta)^2$

Solution: by definition $\alpha + \beta = -b; \alpha\beta = c$. Hence we have;

(i) $\alpha + \beta = -7; \alpha\beta = 4$

(ii) $(\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2 = (\alpha^2 + \beta^2) - 2\alpha\beta \cdots (*)$

But; $(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2 = 49 \therefore \alpha^2 + \beta^2 = 49 - 2\alpha\beta = 49 - 8 = 41$

Hence equation (*) becomes; $(\alpha^2 + \beta^2) - 2\alpha\beta = 41 - 8 = 33$

Exercise

1) Given the pair of roots below determine a corresponding quadratic equation of the form $ax^2 + bx + c = 0$;
   a) (-7, 9)
   b) (-2i, 7)
   c) $\left(\frac{3}{7}, -\frac{8}{9}\right)$
2) Solve the following equations;
   a) $x^2 + 6x - 55 = 0$
   b) $x^2 - 11x + 30 = 0$
   c) $6x^2 - 29x + 35 = 0$
   d) $4x^2 - 29x - 63 = 0$

3) Factorize the following quadratic expressions;
   a) $-3x^2 + 8x + 35$
   b) $77x^2 - 10x - 3$
   c) $8t^2 + 43t + 15$
   d) $7x^2 + 22x + 3$
4) Given that α and $\beta$are the roots of the equation $3x^2 + 4x + 1 = 0$ find the values of;
   a) $(\alpha - \beta)^2$
   b) $\alpha^3 - \beta^3$
   c) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

Bibliography

Antony, C., & Robert, D. (2006). *Foundation Maths*. Prentice Hall.

Kahenya, P. (2017). *Foundation Maths*. LAP Lambert Academic Publishers.

Murray, S., & Robert, M. (2009). *College Algebra*. McGraw-Hill.