Basic Mathematics

Lecture 7

Solving quadratic equations: Completing the square method and Formula

Lecturer: Kahenya, N.P

Introduction to lecture 7

This lecture discusses how to solve quadratic equations using the completing the square method and the quadratic formula.

Intended learning outcomes

At the end of this lecture you will be able to;

(i) Solve quadratic equations using completing the square method.
(ii) Solve quadratic equations using the quadratic formula.

References

These lecture notes should be complemented with relevant topics in (Antony & Robert, 2006;

Kahenya, 2017; Spiegel &Robert, 2009; Seymour, 2020)

Introduction

Factorization method is a weak method. Some quadratic equations cannot be solved using this method especially some whose roots are not integers. In this lecture we look at completing the square method and the quadratic formula. Quadratic formula is a version of completing the square formula for the general equation. This two methods can solve most of the quadratic equations with real variables.

Completing the square method

In this method we work with perfect squares. Perfect squares include 16, 25, 36, 49, 81 etc. Since their roots are whole numbers. Squares such as 10, 12, 24, 38 etc. are not perfect squares since their roots are not whole numbers.

Hence 16, 25, 36, 49 and 81 will give a square of side 4, 5, 6, 7 and 9 respectively.

We can assume the side of a square to be x then we can see that x^2 is a perfect square. If we add 3 to each side of a square of side x we shall get a square of side $(x+3)$ such that $(x+3)^2$ is a perfect square. Expanding it will give x^2+6x+9 (a perfect square).

Example of perfect squares

$(x+5)^2 = x^2 + 10x + 25$ $(3x+5)^2 = 9x^2 + 30x + 25$

$(x-7)^2 = x^2 - 14x + 49$ $(2y-3)^2 = 4y^2 - 12y + 9$

Perfect squares form a quadratic expression of the form $ax^2 + bx + c$. There is a relationship between the three terms of the quadratic expression. One term can be expressed in terms of the other two. In this lecture, we shall consider the third term i.e., constant c

If the expression; $ax^2 + bx + c$ is a perfect square then;

$$c = \frac{b^2}{4a}$$

We can confirm with some examples.

Perfect square	Perfect square expanded	a	b	$c = \frac{b^2}{4a}$
$(x+5)^2$	$x^2 + 10x + 25$	1	10	$c = \frac{b^2}{4a} = \frac{100}{4} = 25$
$(x-7)^2$	$x^2 - 14x + 49$	1	-14	$c = \frac{b^2}{4a} = \frac{(-14)^2}{4} = \frac{196}{4} = 49$
$(3x+5)^2$	$9x^2 + 30x + 25$	9	30	$c = \frac{b^2}{4a} = \frac{900}{36} = 25$
$(2y-3)^2$	$4y^2 - 12y + 9$	4	-12	$c = \frac{b^2}{4a} = \frac{(-12)^2}{16} = \frac{144}{16} = 9$

Note that if a = 1 then we have $c = \frac{b^2}{4} = \left(\frac{b}{2}\right)^2$

Example 1: Determine the value of k to make the following expression a perfect square;

$$x^2 - 12x + k$$

Solution: $k = \left(\frac{b}{2}\right)^2 = \left(\frac{-12}{2}\right)^2 = (-6)^2 = 36$

Example 2: Determine the value of k to make the following expression a perfect square:

$$3x^2 + 7x + k$$

Solution: $k = \frac{b^2}{4a} = \frac{7^2}{4\times3} = \frac{49}{12}$

Alternatively: We first make the coefficient of $x^2 = 1$ i.e. $x^2 + \frac{7}{3}x + \frac{k}{3}$

Thus; $\frac{k}{3} = \left(\frac{b}{2}\right)^2 = \left(\frac{7}{6}\right)^2 = \frac{49}{36} => k = \frac{49}{36} \times 3 = \frac{49}{12}$

Note: The processing of finding the k is referred to as *completing the square.*

Example 3: Solve the following equation by completing the square method; $x^2 + 8x + 12 = 0$

Solution: First take the constant to the RHS (in this case 12) to get; $x^2 + 8x = -12$

Next add a k to both sides to get; $x^2 + 8x + k = -12 + k$

Next find k (i.e., complete the square on the LHS); $k = \left(\frac{b}{2}\right)^2 = \left(\frac{8}{2}\right)^2 = (4)^2$

$$x^2 + 8x + (4)^2 = -12 + (4)^2$$

$$(x + 4)^2 = -12 + 16 = 4$$

Find the square root of both sides to get; $x + 4 = \pm 2$

$$x = -4 \pm 2$$

$$x = -6 \text{ or } x = -2$$

Example 4: Solve $7x^2 + 19x - 6 = 0$

Solution: First, we make the coefficient of $x^2 = 1$ i.e. $x^2 + \frac{19}{7}x - \frac{6}{7} = 0$

Then we take the constant to the RHS (in this case $-\frac{6}{7}$) to get; $x^2 + \frac{19}{7}x = \frac{6}{7}$

Next add a k to both sides to get; $x^2 + \frac{19}{7}x + k = \frac{6}{7} + k$

Next find k (i.e., complete the square on the LHS)

$$k = \left(\frac{b}{2}\right)^2 = \left(\frac{19}{14}\right)^2$$

$$x^2 + 8x + \left(\frac{19}{14}\right)^2 = \frac{6}{7} + \left(\frac{19}{14}\right)^2$$

$$\left(x + \frac{19}{14}\right)^2 = \frac{6}{7} + \frac{361}{196} = \frac{168 + 361}{196} = \frac{529}{196}$$

Find the square root of both sides to get;

$$x + \frac{19}{14} = \pm \frac{23}{14}$$

$$x = -\frac{19}{14} \pm \frac{23}{14} = \frac{-42}{14} \text{ or } \frac{4}{14}$$

$$x = -3 \text{ or } \frac{2}{7}$$

Example 5: Solve $x^2 + x + 7 = 0$

Solution: First take the constant to the RHS (in this case 7) to get; $x^2 + x = -7$

Next add a k to both sides to get; $x^2 + x + k = -7 + k$

Next find k (i.e., complete the square on the LHS)

$$k = \left(\frac{b}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

$$x^2 + x + \left(\frac{1}{2}\right)^2 = -7 + \left(\frac{1}{2}\right)^2$$

$$\left(x + \frac{1}{2}\right)^2 = -7 + \frac{1}{4} = \frac{-28 + 1}{4} = \frac{-27}{4}$$

Find the square root of both sides to get;

$$x + \frac{1}{2} = \pm\sqrt{\frac{-27}{4}} = \pm\frac{\sqrt{9 \times -3}}{\sqrt{4}} = \pm\frac{3\sqrt{-3}}{2} = \pm\frac{3}{2}i\sqrt{3}$$

$$x = -\frac{1}{2} \pm \frac{3}{2}i\sqrt{3}$$

$$x = -0.5 + 1.5i\sqrt{3} \text{ or } x = -0.5 - 1.5i\sqrt{3}$$

Note:

(i) $\sqrt{-1} = i -$ imaginary number that is a member of the complex set.

(ii) $\sqrt{3}$ is an irrational number and it is more accurate to leave it that way (as a radical or surd).

Quadratic formula method

We shall start by solving the general quadratic equation $ax^2 + bx + c = 0$ using the completing the square method.

Solution: We first make the coefficient of $x^2 = 1$ to get; $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Next, we take the constant $\frac{c}{a}$ to the RHS to get; $x^2 + \frac{b}{a}x = -\frac{c}{a}$

We next add a k to both sides and complete the square on the LHS i.e.,

$$x^2 + \frac{b}{a}x + k = -\frac{c}{a} + k$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} = \frac{-4ac + b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

Then find the square root of both sides to get;

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

hence, we have; $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$\mathbf{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Note:

(i) The boxed equation is the **Quadratic formula.** We can use this formula to solve any quadratic equation.

(ii) The procedure above is called deriving the quadratic formula.

Example 1: Solve $18x^2 + 13x + 2 = 0$

Solution: The quadratic formula is given as; $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

In our case; a = 18, b = 13, and c = 2. Hence, we substitute accordingly to get;

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-13 \pm \sqrt{13^2 - 4 \times 18 \times 2}}{2 \times 18} = \frac{-13 \pm \sqrt{25}}{36} = \frac{-13 \pm 5}{36}$$

$$x = -\frac{18}{36} = -\frac{1}{2} \text{ or } x = -\frac{8}{36} = -\frac{2}{9}$$

Nature of the roots

Given the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ then $\boxed{b^2 - 4ac}$ is called the **discriminant** and it can tell the nature of the roots of the quadratic equation.

Discriminant	**Nature of the roots**	**Example**	**Roots**
$b^2 - 4ac = 0$	One repeated rational root	Given $x^2 + 4x + 4 = 0$ then $x = \frac{-4 \pm \sqrt{16 - 16}}{2}$	$\{-2, -2\}$
$b^2 - 4ac > 0$ (a perfect square)	Two rational roots	Given $4x^2 + 11x + 6 = 0$ then $x = \frac{-11 \pm \sqrt{121 - 4 \times 4 \times 6}}{8}$	$\left\{-2, -\frac{3}{4}\right\}$
$b^2 - 4ac > 0$ (not a perfect square)	Two irrational roots	Given $x^2 + 6x + 2 = 0$ then $x = \frac{-6 \pm \sqrt{36 - 8}}{2}$	$\{-3 + \sqrt{7}, -3 - \sqrt{7}\}$
$b^2 - 4ac < 0$	Two complex roots	Given $x^2 + x + 1 = 0$ then $x = \frac{-1 \pm \sqrt{-3}}{2}$	$\left\{\frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}\right\}$

Exercise

1) Determine the value of k to make the following expressions perfect squares;
   a) $x^2 - 7x + k$
   b) $7x^2 + 5x + k$
   c) $x^2 + 13x + k$
   d) $5x^2 - 24x + k$
2) Use the discriminant to determine the nature of the roots of the following quadratic equations
   a) $x^2 + 3x + 6 = 0$
   b) $x^2 + 9x + 14 = 0$
   c) $7x^2 + x + 11 = 0$
   d) $5x^2 + 8x + 3 = 0$
3) Solve using the quadratic formula
   a) $x^2 + 5x + 6 = 0$
   b) $x^2 + 7x + 10 = 0$
   c) $6x^2 + 7x - 12 = 0$
   d) $3x^2 + x + 11 = 0$
4) Solve the following using the completing the square method'
   a) $x^2 + 11x + 30 = 0$
   b) $x^2 + 15x + 36 = 0$
   c) $7x^2 + 15x + 8 = 0$
   d) $3x^2 + 11x + 6 = 0$
   e) $x^2 + x + 11 = 0$
5) Thrice the square of a number is 11 more than 5 times the number. Find the number(s).

Bibliography

Antony, C., & Robert, D. (2006). *Foundation Maths*. Prentice Hall.

Kahenya, P. (2017). *Foundation Maths*. LAP Lambert Academic Publishers.

Murray, S., & Robert, M. (2009). *College Algebra*. McGraw-Hill.

Seymour, L. (2020). *Set Theory and Related Topics*. McGraw-Hill.