Basic Mathematics

Lecture 9

Sequences and series

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Introduction to lecture 9

This lecture discusses sequences and series, their definitions, and the summation notation of series.

Intended learning outcomes

At the end of this lecture you will be able to;

(i) Differentiate between a sequence and a series.
(ii) Carry out operations involving sequences and series.

References

These lecture notes should be complemented with relevant topics in (Antony & Robert, 2006; Kahenya, 2017; Sullivan & Miranda, 2019).

Introduction

The concept of sequences and series is important in calculus in understanding functions and limits. Sequences and series occur naturally e.g. Fibonacci sequence. Some patterns in nature are sequences.

Sequences

Definition 1: A sequence is an ordered arrangement of elements or objects. The objects must follow a certain rule or law. For example, 2,4,6,8,10, ...

Definition 2: A sequence is a list of ordered real numbers. The number in the sequence are also referred to as terms, such that a_1 is the first term, a_2 is the second term, and so on and a_n is the n^{th} term of the sequence.

Definition 3: A sequence is a rule that assigns every natural number a unique real number i.e. a function $x: \mathbb{N} \to \mathbb{R}$.

Definition 4: A sequence may be represented by;
(i) Listing of the terms or elements e.g. $\{2, 4, 6, 8, \cdots\}$
(ii) Stating a rule that generates the terms e.g. $\{5n\} = \{5,10,15,20,\cdots\}$
(iii) Recursion whereby one specifies the first few terms e.g. $a_{n+2} = a_3, a_4, a_5, \cdots$ where $n > 0$.

Definition 5: A sequence $\{a_1, a_2, a_3, \cdots, a_n\}$ is denoted $\{a_n\}$.

Definition 6: A sequence can also be defined by giving the formula of the n^{th} term. For example, consider the sequence;

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots \frac{n}{n+1}, \cdots ; n \geq 1$$

This sequence is defined by $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$

Example 1: Determine the formula for the general term of the sequence; $\left\{\frac{2}{3}, -\frac{3}{9}, \frac{4}{27}, -\frac{5}{81}, \cdots\right\}$

Solution: Our terms are; $a_1 = \frac{2}{3}, a_2 = -\frac{3}{9}, a_3 = \frac{4}{27}, a_4 = -\frac{5}{81}, \cdots$

Note that for $n > 0, n \in \mathbb{N}$;

(i) The numerator of the nth term is $n + 1$.
(ii) The denominator is 3^n.
(iii) The signs are alternatively positive and negative i.e. $(-1)^{n-1}$

Therefore our nth term or general term is given as; $a_n = (-1)^{n-1}\left(\frac{n+1}{3^n}\right)$

Hence we can refer this sequence as $\left\{(-1)^{n-1}\left(\frac{n+1}{3^n}\right)\right\}$.

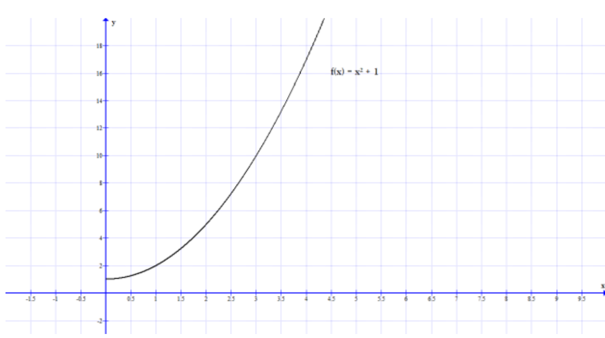
Some sequences are defined recursively and do not have a simple defining equation e.g. Fibonacci sequence $\{f_n\}$.

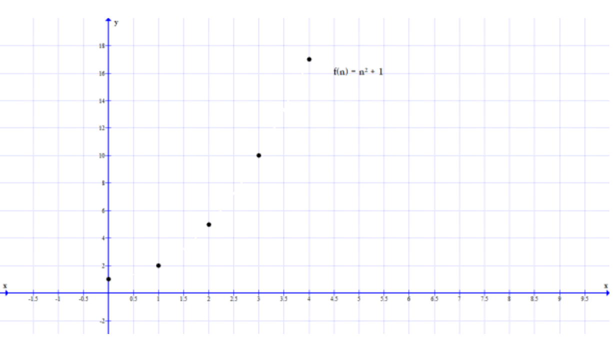
$$f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}, n \geq 3$$

That is; $\{f_n\} = 1,1,2,3,5,8,13,21,34, \cdots$

Definition 7: The elements of a sequence can be plotted to make a graph. There is a close relationship between a sequence and a graph. For example the graph of the function;

$F(x) = x^2 + 1, x \in \mathbb{R}$ and the graph of the sequence $f(n) = n^2 + 1, n \in \mathbb{N}$ are similar as shown in the graphs below.





Definition 8: A sequence may be finite or infinite. A sequence may converge to a fixed point or real number. A sequence may also diverge.

Series

Definition 1: A series is the sum of terms of a sequence. For example, given the sequence 2,4,6,8,10, ... Then 2 + 4 + 6 + 8 +10 + ... is a series.

Definition 2: A finite series (with n terms) maybe denoted;

$$S_n = \sum_{i=1}^{n} x_i = x_1 + x_2 + x_n + \cdots + x_n$$

An infinite series is denoted;

$$S_\infty = \sum_{i=1}^{\infty} x_i = x_1 + x_2 + x_3 + \cdots + x_n + x_{n+1} + \cdots$$

Definition 3: Some series are convergent while others are divergent. Others oscillate.

For example, a series $1 + 2 + 3 + 4 + \cdots$ is a divergent series. It is also increasing.

A series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ is a decreasing series that is converging to a certain fixed number. In fact, this series converges to the real number 2 (this will be demonstrated in lecture 10).

Sequences occurs naturally. Seasons, heat wave, times etc. form sequences or series.

Series are also used in say loan repayments, to understand growth pattern of population of species, decaying of radioactive elements among others.

Example 1:

Evaluate the series;

$$\sum_{i=0}^{4} (2^i+3i)$$

Solution:

$$\sum_{i=0}^{4} (2^i + 3i) = (2^0 + 3 \cdot 0 +) + (2^1 + 3 \cdot 1) + (2^2 + 3 \cdot 2) + (2^3 + 3 \cdot 3) + (2^4 + 3 \cdot 4)$$

$$= 1 + 5 + 10 + 17 + 28 = 61$$

Example 2:

Given $x_1 = 1, x_2 = -1, x_3 = 3, x_4 = -4$ evaluate;

$$\sum_{i=1}^{4} \left(\frac{x_i^2 + x_i}{-x_i}\right)$$

Solution:

$$\sum_{i=1}^{4} \left(\frac{x_i^2 + x_i}{-x_i}\right) = \left(\frac{x_1^2 + x_1}{-x_1}\right) + \left(\frac{x_2^2 + x_2}{-x_2}\right) + \left(\frac{x_3^2 + x_3}{-x_3}\right) + \left(\frac{x_4^2 + x_4}{-x_4}\right)$$

$$= \left(\frac{1^2 + 1}{-1}\right) + \left(\frac{1 - 1}{1}\right) + \left(\frac{3^2 + 3}{-3}\right) + \left(\frac{16 - 4}{4}\right) = -2 + 0 - \frac{12}{3} + \frac{12}{4} = -3$$

Example 3:

Expand the sequence;

$$\{(-1)^n\}, n \in \mathbb{N}$$

Solution:

$$\{(-1)^n\} = \cdots - 1, 1, -1, 1, -1, \cdots, (-1)^n, \cdots$$

This sequence does not approach any limit and therefore we say it is divergent.

Example 4: Expand the sequence;

$$\left\{(-1)^n \frac{1}{2n - 1}\right\}, n \in \mathbb{N}$$

Solution:

$$\left\{(-1)^n \frac{1}{2n - 1}\right\} = -1, \frac{1}{3}, -\frac{1}{5}, \frac{1}{7}, -\frac{1}{9}, \frac{1}{11}, \cdots, \left((-1)^n \frac{1}{2n - 1}\right), \cdots$$

Example 5: Find the first 6 terms of the sequence

$$\left\{\frac{n}{n^2 + 5}\right\}$$

Solution:

$$\left\{\frac{n}{n^2 + 5}\right\} = 0, \frac{1}{6}, \frac{2}{9}, \frac{3}{14}, \frac{4}{21}, \frac{5}{30}$$

We can also find the sum of the first 6 terms of the series;

$$\sum_{n=0}^{5} \frac{n}{n^2 + 5} = 0 + \frac{1}{6} + \frac{2}{9} + \frac{3}{14} + \frac{4}{21} + \frac{5}{30} = \frac{121}{126}$$

Example 6: Write down the first 5 terms of the series;

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)} = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \cdots$$

Example 7: Expand $n!$ (factorial n) and hence evaluate 6!

Solution: $n! = n(n-1)(n-2)(n-2)\cdots 4\cdot 3\cdot 2\cdot 1$

Evaluate 6! = $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

Example 8: Write down the first 6 terms of the sequence;

$$\left\{\frac{n!}{3^n}\right\}$$

Solution:

$$1! = 1$$

$$2! = 2\cdot 1 = 2$$

$$3! = 3\cdot 2\cdot 1 = 6$$

$$4! = 4\cdot 3\cdot 2\cdot 1 = 24$$

$$5! = 5\ \cdot 4\cdot 3\cdot 2\cdot 1 = 120$$

$$6! = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 720$$

$$\therefore \left\{\frac{n!}{3^n}\right\} = \frac{1}{3}, \frac{2}{9}, \frac{6}{27}, \frac{24}{81}, \frac{120}{243}, \frac{720}{729}$$

Note that $0! = 1$

Exercise

1) Determine the next three terms in the following sequences
   a) 13, 16, 19, 22
   b) -4, -6, -8,
   c) 13, 6.5, 3.25, 1.625
   d) 0.5, 0.05, 0.005
2) Given that $x_1 = 2, x_2 = 3, x_3 = 4, x_4 = -3, x_5 = -1$ evaluate the series below.

a) $\sum_{i=1}^{5}(x_i + x_i^2 + 1)$ b) $\sum_{i=1}^{5}\frac{6x_i}{x_i^3}$ c) $\sum_{i=1}^{5}\left(\frac{x_i}{5 - x_i^2}\right)$

3) Determine the first 5 terms on the sequences below;

a) $\left\{\frac{n}{n^2+1}\right\}_{n=1}^{\infty}$

b) $\left\{\frac{1}{3n+1}\right\}, n > 0$

c) $\{3n^2 + 2n - 1\}, n > 0$

d) $\{(n+1)(n+2)\}, n > 0$

4) Evaluate;

$$\sum_{k=0}^{5}(3^k - 8 + 5^{3k})$$

5) Evaluate;

$$\sum_{k=0}^{7}\frac{k+5}{k^2+6k+4}$$

6) Evaluate;

$$\sum_{k=1}^{5}\frac{7}{4^{k+1}}$$

Bibliography

Antony, C., & Robert, D. (2006). *Foundation Maths*. Prentice Hall.

Kahenya, P. (2017). *Foundation Maths*. LAP Lambert Academic Publishers.

Sullivan, M., & Miranda, K. (2019). *Calculus: Early Transcendentals* (second). W.H. Freeman and Company.