Basic Mathematics

Lecture 10

Arithmetic and Geometric Progressions

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Introduction to lecture 10

This lecture discusses sequences and series, and in particular arithmetic and geometric progressions. These are popular series.

Intended learning outcomes

At the end of this lecture you will be able to;

(i) Differentiate between GP and AP.
(ii) Carry out operations involving AP and GP.

References

These lecture notes should be complemented with relevant topics in (Antony & Robert, 2006; Kahenya, 2017; Sullivan & Miranda, 2019).

Arithmetic progression

Arithmetic Progression AP is a sequence which proceeds with constant difference (common difference), d. For example: 4, 9, 14, 19, 24, … is an AP with common difference d = 5, and the first term $a = T_1 = 4$

In general, an A.P with n terms is of the form;

$$a, [a + d], [a + 2d], [a + 3d], \cdots, [a + (n - 1)d]$$

Where $[a + (n - 1)d]$ is the n^{th} term $= T_n$

Example 1

Given the AP; 3, 10, 17, 24, 31… find the 30^{th} term in the sequence.

Solution: By definition, $T_n = a + (n - 1)d$. Note our;

$$a = 3;\ n = 30;\ d = T_n - T_{n-1} = T_2 - T_1 = 10 - 3 = 7 \text{ (or } T_5 - T_4 = 31 - 24 = 7)$$

Hence; $T_{30} = 3 + (30 - 1)7 = 3 + 29 \times 7 = 206$

Example 2

Nimo was repaying a loan in instalments. The instalments for the first five months were as follows; 11500, 11200, 10900, 10600, 10300, …Find her 15th instalment.

Solution: Note our first term, a = 11500, Number of terms, n = 15, and our common difference d = 11200 – 11500 = -300

Hence; $T_n = a + (n-1)d \Rightarrow T_{15} = 11500 + (15-1)(-300) = 11500 - 4200 = 7300$

Sum of the first n terms of an A.P

To get the sum of n terms of an AP, S_n consider the series;

$$S_n = a + (a+d) + (a+2d) + (a+3d) + \cdots + (l-d) + l \cdots \text{(i)} \text{where } l = T_n$$

We can also rewrite this series as;

$$S_n = l + (l-d) + (l-2d) + \cdots + (a+3d) + (a+2d) + (a+d) + a \cdots \text{(ii)}$$

Add equations (ii) and (i) to get;

$$\begin{aligned} S_n &= a + (a+d) + (a+2d) + \cdots + (l-2d) + (l-d) + l \\ S_n &= l + (l-d) + (l-2d) + \cdots + (a+2d) + (a+d) + a \\ \hline 2S_n &= (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l) + (a+l) \end{aligned}$$

This shows that; $2S_n = n(a+l)$

$$\mathbf{S_n = \frac{n}{2}(a+l) \cdots (*)}$$

NB: This formula is useful especially when you know the last term l of the given AP.

Otherwise, since $l = T_n = a + (n-1)d$ we can replace l in equation (*) to get;

$$S_n = \frac{n}{2}[a + (a + (n-1)d]$$

$$\mathbf{S_n = \frac{n}{2}[2a + (n-1)d] \cdots (**)}$$

Example 3

Given the series; 12 + 18 + 24 + 30 + 36 determine the 100th term and the sum of the first 100 terms.

Solution: Our a = 12, n = 100, common difference d = 18 – 12 = 6

Hence; $T_{100} = a + (n-1)d = 12 + (100-1)6 = 12 + 99 \times 6 = 606$

$$S_{100} = \frac{n}{2}(a + l) = \frac{100}{2}(12 + 606) = 50(618) = 30900$$

Alternatively; $S_{100} = \frac{n}{2}[a + (n-1)d] = \frac{100}{2}[12 + (100-1)6] = 50(12 + 99 \times 6) = 30900$

Geometric Progression

Geometric sequence is a sequence of the form; $a, ar, ar^2, ar^3, \cdots, ar^{n-1}$ where a is the first term, r is the common ratio and the nth term;

$$T_n = ar^{n-1}$$

Note that the common ratio

$$r = \frac{ar}{a} = \frac{ar^{n-1}}{ar^{n-2}} = \frac{ar^n}{ar^{n-1}}$$

Example 1

(i) 3, 9, 27, 81, … is a geometric sequence where a = 3 and $r = \frac{9}{3} = \frac{27}{9} = \frac{81}{27} = 3$

(ii) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$ is a geometric sequence where a = 1 and $r = \frac{1}{2}$

Example 2

Given the sequence; $2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \cdots$ Find the common ratio r, the 10th term, and the 100th term.

Solution: The comon ratio $r = \frac{T_2}{T_1} = \frac{\frac{2}{3}}{2} = \frac{1}{3}$.

Hence, the 10^{th} term $= T_{10} = ar^9 = 2 \times \left(\frac{1}{3}\right)^9 = \frac{2}{3^9}$

Next, the 100th term, $T_{100} = 2 \times \left(\frac{1}{3}\right)^{99} = \frac{2}{3^{99}}$

Sum of first n terms of a GP

Consider the sum of the first n terms of a geometric series;

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} \cdots \text{(i)}$$

Next, we multiply both sides of (i) by the common ratio r to get;

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n \cdots \text{(ii)}$$

We have two options;

(a) We subtract equation (ii) from equation (i) or

(b) We subtract equation (i) from equation (ii).

We shall end up with two equations that we use to find the sum of the first n terms of a GP.

Option (a): (We subtract equation (ii) from equation (i))

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$
$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n$$
$$S_n - rS_n = (a - ar) + (ar - ar^2) + (ar^2 - ar^3) + \cdots + (ar^{n-1} - ar^n)$$

$$S_n - rS_n = a - ar + ar - ar^2 + ar^2 - ar^3 + \cdots + ar^{n-1} - ar^n$$

Simplifying the resultant, we have;

$S_n(1 - r) = a - ar^n$

$$\mathbf{S_n = \frac{a(1 - r^n)}{1 - r}; \text{used when } |r| < 1}$$

This formula is appropriate when the common ratio is less than one.

Example 3

Find the sum of the first 7 terms of the series $5 + \frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \cdots$

Solution: The common ratio $r = \frac{1}{2}$ (which is less than 1).

Hence, we have;

$$S_7 = \frac{a(1 - r^n)}{1 - r} = \frac{5(1 - 0.5^7)}{1 - 0.5} = \frac{5\left(1 - \frac{1}{2^7}\right)}{\frac{1}{2}} = 10\left(1 - \frac{1}{2^7}\right) = \frac{635}{64} = 9.921875$$

Option (b): (We subtract equation (i) from equation (ii))

$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n$

$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$

$rS_n - S_n = (ar - a) + (ar^2 - ar) + (ar^3 - ar^2) + \cdots + (ar^n - ar^{n-1})$

Simplifying the resultant, we have;

$$S_n(r-1) = ar^n - a$$

$$\mathbf{S_n = \frac{a(r^n - 1)}{r - 1}; \text{used when } |r| > 1}$$

Example 4: Find the sum of the first 10 terms of the series 5 + 15+ 45+ 135+…

Solution: Note that our a = 5, n = 10, and common ratio $r = \frac{15}{5} = 3$

$$S_{10} = \frac{a(r^n - 1)}{r - 1} = \frac{5(3^{10} - 1)}{3 - 1} = 2.5(3^{10} - 1) = 147620$$

Example 5: A man decide to save a shilling on day 1. On day 2 saved 2 shillings, on day 3 he saved 4 shillings, on day 4 he saved 8 shillings, and so forth. How much will he save on the 20th day? What will be his total savings by end of day 20?

Solution: The savings give us the series; $1 + 2 + 4 + 8 + \cdots$ with first term $a = 1$, and the common ratio $r = 2$. Then we have;

How much will he save on the 20th day i.e. T_{20}?

$$T_{20} = ar^{19} = 1 \times 2^{19} = 524{,}288 \text{ shillings}$$

What will be his total savings by end of day 20 i.e. S_{20}?

$$S_{20} = \frac{a(r^n - 1)}{r - 1} = \frac{1(2^{20} - 1)}{2 - 1} = 2^{20} - 1 = 1{,}048{,}575 \text{ shillings}$$

Sum to infinity of a GP that is converging

This is the case where the progression is up to infinity and the progression is also decreasing. The common ratio r is normally $0 < r < 1$. Hence, we can use the formula;

$$S_n = \frac{a(1-r^n)}{1-r}$$

The problem is that this is for finite series. Hence, we find the limit as n approaches infinity i.e.

$$S_\infty = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{a(1-r^n)}{1-r}\right) = \frac{a}{1-r}\text{, since } r^n \to 0 \text{ as } n \to \infty$$

Thus;

$$\mathbf{S_\infty = \frac{a}{1-r}}$$

Note that if $|r| \geq 1$, then sum to infinity is infinity i.e. $S_\infty = \infty$

Example 5

Consider the geometric progression; $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ the series proceeds to infinity.

Solution: In our case a = 1 and common ratio $r = \frac{1}{2}$ then,

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$$

Example 6

Consider the recurring decimal; $0.777\cdots$ Convert it into a fraction.

Solution: The recurring decimal can also be written as;

$$0.777\cdots = 0.7 + 0.07 + 0.007 + 0.0007 + \cdots$$

Note that this is a series and in particular a GP that progresses to infinity. Our first term $a = 0.7$ and the common ratio; $r = \frac{0.07}{0.7} = 0.1$

Hence we have; $S_\infty = \frac{a}{1-r} = \frac{0.7}{1-0.1} = \frac{0.7}{0.9} = \frac{7}{9}$

Exercise

1. The first term of an A.P is 9 and the sum of the first 20 terms is 2270. Find the n^{th} term, and the 10^{th} term
2. Find the sum of the first 50 terms of the A.P; 110+ 117+124+131+…
3. The first and the n^{th} terms of an A.P are 11 and 606, and the sum of the n terms is 11106. Find the number of terms, n and the common difference, d.
4. The 2^{nd} term of an A.P is 28 and the 11^{th} term - 53. Find the 1^{st} term, the common difference, and the sum of the first 29 terms.
5. Karis' first month commission is KES 14500 and increase by KES 230 a month. How much will earn in the 9^{th} month? How much would he have earned in 15^{th} months?
6. Determine the next five terms in the following sequences
   a) 7, 14, 28,56
   b) $\frac{1}{7}, \frac{1}{35}, \frac{1}{175},$
7. Calculate the sum to infinity of the following GPs
   a) $\frac{2}{3} + \frac{1}{2} + \frac{3}{8} + \frac{9}{32} + \cdots$
   b) $2 + 6 + 18 + \cdots$
   c) $0.12121212\cdots$
   d) $0.9 + 0.09 + 0.009 + \cdots$
8. Three consecutive terms of a geometric series are the 2^{nd}, 6^{th}, and 14^{th} terms of arithmetic series. Find the common ratio of the geometric series.

Bibliography

Antony, C., & Robert, D. (2006). *Foundation Maths*. Prentice Hall.

Kahenya, P. (2017). *Foundation Maths*. LAP Lambert Academic Publishers.

Sullivan, M., & Miranda, K. (2019). *Calculus: Early Transcendentals* (second). W.H. Freeman and Company.