Classical Mechanics

4 Rigid body dynamics

In this section we discuss the extension of Lagrangian mechanics for point particles to cover the dynamics of extended bodies. A *rigid body* may be defined as any distribution of mass for which the distance between any two points is fixed. For example, we can consider a finite number of particles with position vectors $\mathbf{r}_I$ $(I = 1, \ldots, N)$ as a rigid body, provided that we impose the constraints $|\mathbf{r}_I - \mathbf{r}_J| = c_{IJ} =$ constant. One might imagine these as the positions of atoms in a solid, with the constraints arising from inter-molecular forces. However, we will more often model a rigid body as a continuous distribution of matter, which may be regarded as a limit of the point particle model in which the number of particles tends to infinity. However, before getting into the details of this, we first need to give a more precise description of rotating frames.

4.1 Rotating frames and angular velocity

Following on from our discussion in section 1.1, consider two reference frames $\mathcal{S}$, $\hat{\mathcal{S}}$, both with the same origin O. Focusing first on the frame $\mathcal{S}$, its coordinate axes correspond to three orthonormal vectors $\mathbf{e}_i$, $i = 1, 2, 3$. The $\{\mathbf{e}_i\}$ form a basis for $\mathbb{R}^3$, and being orthonormal means they have inner products $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ (the *Kronecker delta symbol* δ_{ij} is defined in the appendix). We may then write a position vector as $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = (x_1, x_2, x_3)$ with respect to this basis, where $x_i = \mathbf{r} \cdot \mathbf{e}_i$ are the *components of* $\mathbf{r}$ *in the frame* $\mathcal{S}$.

The frame $\hat{\mathcal{S}}$ will similarly have an orthonormal basis $\{\hat{\mathbf{e}}_i\}$, and we may write

$$\hat{\mathbf{e}}_i = \sum_{j=1}^{3} \mathcal{R}_{ji}\mathbf{e}_j \ . \tag{4.1}$$

Taking the dot product with $\mathbf{e}_k$, this is equivalent to defining $\mathcal{R}_{ki} = \mathbf{e}_k \cdot \hat{\mathbf{e}}_i$. The position vector $\mathbf{r}$ then has an expansion $\mathbf{r} = \hat{x}_1\hat{\mathbf{e}}_1 + \hat{x}_2\hat{\mathbf{e}}_2 + \hat{x}_3\hat{\mathbf{e}}_3 = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ in this basis. In section 1.1 we would have referred to this as $\hat{\mathbf{r}}$, as we wanted to emphasize that it is the components $\hat{x}_i = \mathbf{r} \cdot \hat{\mathbf{e}}_i$ that we measure in the frame $\hat{\mathcal{S}}$. However, mathematically $\mathbf{r}$ and $\hat{\mathbf{r}}$ are the same vector, just expressed in different bases. The *components* of the vector are related as

$$\hat{x}_i = \mathbf{r} \cdot \hat{\mathbf{e}}_i = \sum_{j=1}^{3} \mathcal{R}_{ji}(\mathbf{r} \cdot \mathbf{e}_j) = \sum_{j=1}^{3} \mathcal{R}_{ji}x_j \ . \tag{4.2}$$

The matrix $\mathcal{R}$ is *orthogonal*, as one sees by taking the dot product of (4.1) with $\hat{\mathbf{e}}_k$:

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_k = \delta_{ik} = \sum_{j=1}^{3} \mathcal{R}_{ji}\mathcal{R}_{jk} \ . \tag{4.3}$$

Thus $\mathbb{1} = \mathcal{R}^T\mathcal{R} = \mathcal{R}\mathcal{R}^T$, where $\mathbb{1}$ denotes the 3×3 identity matrix. The matrices in $GL(3, \mathbb{R})$ satisfying this condition form a subgroup called the *orthogonal group* $O(3)$. Notice that $\mathcal{R} \in O(3)$ implies that $\det \mathcal{R} = \pm 1$. Using the definition of the scalar triple product as a determinant it is straightforward to check that $\hat{\mathbf{e}}_1 \cdot (\hat{\mathbf{e}}_2 \wedge \hat{\mathbf{e}}_3) = (\det \mathcal{R})\, \mathbf{e}_1 \cdot (\mathbf{e}_2 \wedge \mathbf{e}_3)$. Since $\{\mathbf{e}_i\}$ is orthonormal we must have $\mathbf{e}_1 \cdot (\mathbf{e}_2 \wedge \mathbf{e}_3) = \pm 1$, and we shall always choose a *right-handed frame* in which $\mathbf{e}_1 \cdot (\mathbf{e}_2 \wedge \mathbf{e}_3) = +1$. In order that a rotation maps a right-handed frame to another right-handed frame, we thus have $\det \mathcal{R} = 1$. The subgroup of $O(3)$ satisfying this condition is called the *special orthogonal group* $SO(3)$. (The elements in $O(3)$ that are not in $SO(3)$ involve a reflection.)

In general the rotation matrix $\mathcal{R} = \mathcal{R}(t)$ will depend on time t, so that the coordinate axes of $\mathcal{S}$ are rotating relative to those of $\hat{\mathcal{S}}$ (and *vice versa*). Differentiating the orthogonality relation $\mathcal{R}\mathcal{R}^T = \mathbb{1}$ with respect to t we obtain

$$\dot{\mathcal{R}}\mathcal{R}^T + \mathcal{R}\dot{\mathcal{R}}^T \;=\; 0\,, \tag{4.4}$$

which since $\mathcal{R}\dot{\mathcal{R}}^T = (\dot{\mathcal{R}}\mathcal{R}^T)^T$ implies that the matrix

$$\Omega \;\equiv\; \mathcal{R}\dot{\mathcal{R}}^T \;=\; -\dot{\mathcal{R}}\mathcal{R}^T \;=\; \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}\,, \tag{4.5}$$

is anti-symmetric. In the last equality we have introduced functions $\omega_i = \omega_i(t)$, $i = 1, 2, 3$, parametrizing the non-zero entries of this matrix. Notice that we may rewrite (4.5) as

$$\Omega_{ij} \;=\; -\sum_{k=1}^{3} \epsilon_{ijk}\omega_k\,, \tag{4.6}$$

using the *Levi-Civita alternating symbol* ϵ_{ijk} defined in the appendix. The *angular velocity* of the frame $\mathcal{S}$ relative to $\hat{\mathcal{S}}$ is then defined to be the vector $\boldsymbol{\omega} = \sum_{i=1}^{3} \omega_i \mathbf{e}_i$. In general this is a function of time $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$.

Now we come to the main point of this subsection. In section 1.1 we simply defined $\dot{\mathbf{r}}$ by differentiating its components in the frame. However, if one applies this definition in two different frames which are rotating relative to each other, then the resulting vector will depend on the choice of frame. In this situation the notation "$\dot{\mathbf{r}}$" is hence too ambiguous. In order to be clear, we will therefore define the *time derivative* of a vector $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ with respect to the orthonormal basis $\{\mathbf{e}_i\}$ of a frame $\mathcal{S}$ to be the vector

$$\mathrm{D}\mathbf{r} \;\equiv\; \dot{x}_1\mathbf{e}_1 + \dot{x}_2\mathbf{e}_2 + \dot{x}_3\mathbf{e}_3\,. \tag{4.7}$$

For example, if $\mathbf{r}(t)$ is the position vector of a particle from the origin of $\mathcal{S}$, then $\mathrm{D}\mathbf{r}$ and $\mathrm{D}^2\mathbf{r}$ are its velocity and acceleration relative to $\mathcal{S}$. We may then derive

The Coriolis formula: The time derivatives $\mathrm{D}\mathbf{r}$ and $\hat{\mathrm{D}}\mathbf{r}$ of $\mathbf{r}$ relative to the two frames $\mathcal{S}$ and $\hat{\mathcal{S}}$ are related by

$$\hat{\mathrm{D}}\mathbf{r} \;=\; \mathrm{D}\mathbf{r} + \boldsymbol{\omega} \wedge \mathbf{r}\,, \tag{4.8}$$

where $\boldsymbol{\omega}$ is the angular velocity of $\mathcal{S}$ relative to $\hat{\mathcal{S}}$.

The proof is just a direct computation. Taking the time derivative of (4.2), in vector notation we have

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{pmatrix} = \mathcal{R}^T \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} + \dot{\mathcal{R}}^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} . \tag{4.9}$$

Multiplying on the left by $\mathcal{R}$ this reads

$$\mathcal{R} \begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} + \Omega \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} . \tag{4.10}$$

Given the formula (4.5), the components of the second term on the right hand side are precisely the components of $\boldsymbol{\omega} \wedge \mathbf{r}$ in the basis $\{\mathbf{e}_i\}$.[10] The components of $\hat{\mathrm{D}}\mathbf{r}$ in the basis $\{\hat{\mathbf{e}}_i\}$ for the frame $\hat{\mathcal{S}}$ are $\dot{\hat{x}}_1$, $\dot{\hat{x}}_2$, $\dot{\hat{x}}_3$, so the entries of the column vector on the left hand side of (4.10) are the components of $\hat{\mathrm{D}}\mathbf{r}$ in basis $\{\mathbf{e}_i\}$ for the frame $\mathcal{S}$. This proves (4.8).

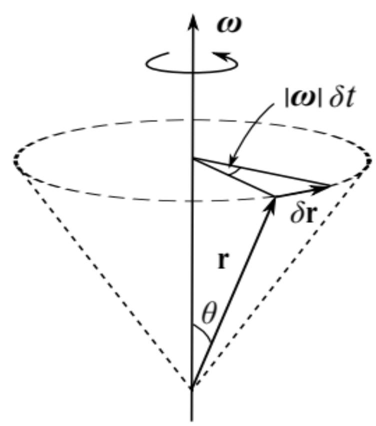

Figure 10: As seen in the frame $\hat{\mathcal{S}}$, the position vector $\mathbf{r}$ of a point P fixed in the frame $\mathcal{S}$ changes by $\delta\mathbf{r} = \boldsymbol{\omega} \wedge \mathbf{r}\,\delta t$ in a small time interval δt. This is a rotation of $\mathbf{r}$ through an angle $|\boldsymbol{\omega}|\delta t$ about an axis parallel to the vector $\boldsymbol{\omega}$.

In the application to rigid bodies, we will be interested in applying this formula to the case where $\hat{\mathcal{S}}$ is an inertial frame (which we regard as the frame in which we are viewing the motion), while the frame $\mathcal{S}$ rotates with the rigid body; that is, $\mathcal{S}$ is the *rest frame of the body*, which in general will not be an inertial frame. By definition any point P in the body is then at rest in $\mathcal{S}$, so its position vector $\mathbf{r}$ has $\mathrm{D}\mathbf{r} = \mathbf{0}$. But the time derivative of $\mathbf{r}$ in the inertial frame $\hat{\mathcal{S}}$ is given by

$$\hat{\mathrm{D}}\mathbf{r} = \boldsymbol{\omega} \wedge \mathbf{r} , \tag{4.11}$$

where $\boldsymbol{\omega}$ is the angular velocity of the rest frame of the rigid body. The formula (4.11) determines the way in which the vector $\mathbf{r}$ changes relative to the frame $\hat{\mathcal{S}}$. To get some geometric intuition for

[10] The computation is $(\Omega\,\mathbf{r})_i = \sum_{j=1}^3 \Omega_{ij} x_j = -\sum_{j,k=1}^3 \epsilon_{ijk}\omega_k x_j = \sum_{j,k=1}^3 \epsilon_{ijk}\omega_j x_k = (\boldsymbol{\omega} \wedge \mathbf{r})_i$, using (4.6).

this, consider the change $\delta\mathbf{r}$ in $\mathbf{r}$ in a small time interval δt, where we ignore quadratic and higher order terms. This is $\delta\mathbf{r} = \boldsymbol{\omega}\wedge\mathbf{r}\,\delta t$. This vector is orthogonal to both $\boldsymbol{\omega}$ and $\mathbf{r}$, and has modulus $|\boldsymbol{\omega}||\mathbf{r}|\sin\theta\,\delta t$, where θ is the angle between $\boldsymbol{\omega}$ and $\mathbf{r}$ – see Figure 10. As seen in $\hat{\mathcal{S}}$, the change in the position vector $\mathbf{r}$ of the point P fixed in the body in the time interval δt is hence obtained by rotating $\mathbf{r}$ through an angle $|\boldsymbol{\omega}|\delta t$ about an axis parallel to $\boldsymbol{\omega}$. The direction of $\boldsymbol{\omega}$ is thus the *instantaneous axis of rotation*, while its magnitude $|\boldsymbol{\omega}|$ is the rate of rotation.

Example: Consider the special case in which

$$\mathcal{R}(t) \;=\; \begin{pmatrix} \cos\varphi(t) & \sin\varphi(t) & 0 \\ -\sin\varphi(t) & \cos\varphi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}\,, \qquad (4.12)$$

so that $\mathbf{e}_1 = \cos\varphi(t)\,\hat{\mathbf{e}}_1 + \sin\varphi(t)\,\hat{\mathbf{e}}_2$, $\mathbf{e}_2 = -\sin\varphi(t)\,\hat{\mathbf{e}}_1 + \cos\varphi(t)\,\hat{\mathbf{e}}_2$, $\mathbf{e}_3 = \hat{\mathbf{e}}_3$ is a rotation about the third axis. Then one computes $\boldsymbol{\omega} = \dot{\varphi}\,\mathbf{e}_3$.

As a final comment in this subsection, we note that $\boldsymbol{\omega}$ behaves as you would expect a vector to do. For example, if the angular velocity of $\mathcal{S}$ relative to $\hat{\mathcal{S}}$ is $\boldsymbol{\omega}$, then the angular velocity of $\hat{\mathcal{S}}$ relative to $\mathcal{S}$ is $-\boldsymbol{\omega}$. Moreover, if $\hat{\mathcal{S}}$ in turn has angular velocity $\hat{\boldsymbol{\omega}}$ relative to another frame $\mathcal{S}'$, then $\mathcal{S}$ has angular velocity $\boldsymbol{\omega}+\hat{\boldsymbol{\omega}}$ relative to $\mathcal{S}'$. One can prove these statements either by using the Coriolis formula (4.8) to characterize $\boldsymbol{\omega}$, or else by introducing rotation matrices for each change of basis and using the definition (4.5)

4.2 Motion in a non-inertial frame

In this subsection we take a brief detour from our main topic, and present Newton's equations in a general non-inertial frame.

Suppose that $\hat{\mathcal{S}}$ is an inertial frame with origin $\hat{O}$, and $\mathcal{S}$ is another frame whose origin O is at position vector $\mathbf{x} = \mathbf{x}(t)$ from $\hat{O}$. If $\mathbf{r}$ denotes the position vector of a particle measured from O, and $\hat{\mathbf{r}}$ is the position of the particle measured from $\hat{O}$, then

$$\hat{\mathbf{r}} \;=\; \mathbf{r}+\mathbf{x}\,. \qquad (4.13)$$

As in the previous subsection we define the acceleration of the particle relative to $\mathcal{S}$ as $\mathbf{a} = \mathrm{D}^2\mathbf{r}$, and similarly relative to $\hat{\mathcal{S}}$ we have $\hat{\mathbf{a}} = \hat{\mathrm{D}}^2\hat{\mathbf{r}}$. Taking $\hat{\mathrm{D}}^2$ of (4.13) and using the Coriolis formula (4.8) we obtain

$$\begin{aligned} \hat{\mathbf{a}} &= \hat{\mathrm{D}}^2(\mathbf{r}+\mathbf{x}) \;=\; \hat{\mathrm{D}}(\mathrm{D}\mathbf{r}+\boldsymbol{\omega}\wedge\mathbf{r})+\mathbf{A} \\ &= \mathbf{a}+(\mathrm{D}\boldsymbol{\omega})\wedge\mathbf{r}+2\boldsymbol{\omega}\wedge\mathrm{D}\mathbf{r}+\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r})+\mathbf{A}\,. \end{aligned} \qquad (4.14)$$

Here we have defined $\mathbf{A} = \hat{\mathrm{D}}^2\mathbf{x}$, which is the acceleration of O relative to $\hat{\mathcal{S}}$, and it is simple enough to check from the definition that $\mathrm{D}(\mathbf{b}\wedge\mathbf{c}) = (\mathrm{D}\mathbf{b})\wedge\mathbf{c}+\mathbf{b}\wedge\mathrm{D}\mathbf{c}$ for any two vectors $\mathbf{b}$, $\mathbf{c}$.

Since $\hat{\mathcal{S}}$ is an inertial frame, Newton's second law holds and for a particle of mass m we may write

$$m\hat{\mathbf{a}} \;=\; \mathbf{F} \,, \tag{4.15}$$

where $\mathbf{F}$ is the force acting. Substituting from (4.14) we thus have

$$m\mathbf{a} \;=\; \mathbf{F} - m(\mathrm{D}\boldsymbol{\omega}) \wedge \mathbf{r} - 2m\boldsymbol{\omega} \wedge \mathrm{D}\mathbf{r} - m\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r}) - m\mathbf{A} \,. \tag{4.16}$$

The additional terms on the right hand side of (4.16) may be interpreted as "ficticious forces":

$$\begin{aligned} \mathbf{F}_1 &= -m(\mathrm{D}\boldsymbol{\omega}) \wedge \mathbf{r} \,, & \mathbf{F}_2 &= -2m\boldsymbol{\omega} \wedge \mathrm{D}\mathbf{r} \,, \\ \mathbf{F}_3 &= -m\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r}) \,, & \mathbf{F}_4 &= -m\mathbf{A} \,. \end{aligned} \tag{4.17}$$

That is, these may be regarded as corrections to the force in $\mathbf{F} = m\mathbf{a}$ due to the fact that the frame $\mathcal{S}$ is accelerating. The force $\mathbf{F}_1$ is known as the *Euler force*, and arises from the *angular acceleration* of $\mathcal{S}$. The Euler force is hence zero for a frame rotating at constant angular velocity, $\mathrm{D}\boldsymbol{\omega} = \mathbf{0}$. The force $\mathbf{F}_2$ is known as the *Coriolis force*, and is interesting in that it depends on the velocity $\mathrm{D}\mathbf{r}$ of the particle as measured in $\mathcal{S}$. The Coriolis force plays a role in weather, for example being responsible for the circulation of air around an area of low pressure, which is anti-clockwise in the northern hemisphere. The force $\mathbf{F}_3$ is the *centrifugal force*. It lies in a plane through $\mathbf{r}$ and $\boldsymbol{\omega}$, is perpendicular to the axis of rotation $\boldsymbol{\omega}$, and is directed away from the axis. This is the force you experience standing on a roundabout. Finally, $\mathbf{F}_4$ is simply due to the acceleration of the origin O. For example, this force effectively cancels the Earth's gravitational field in a freely-falling frame.

4.3 Rigid body motion and the inertia tensor

We turn now to our main topic for this section, namely rigid body motion. From the outset we fix a reference inertial frame $\hat{\mathcal{S}}$ with origin $\hat{O}$. A particular point in the body then has position vector $\mathbf{x} = \mathbf{x}(t)$ relative to $\hat{O}$. We denote this point by O, and take it to be the origin of the rest frame $\mathcal{S}$ of the body. Provided the matter distribution is not all along a line, this rest frame is defined uniquely by the body, up to a *constant* orthogonal rotation of its axes and a translation of the origin by a *constant* vector (relative to $\mathcal{S}$). In particular this latter freedom just shifts the choice of fixed point in the body. As we shall see shortly, the body itself provides a natural way to fix both of these freedoms.

As in section 4.1, the rotation of the frame $\mathcal{S}$ relative to the inertial frame $\hat{\mathcal{S}}$ is determined by the time-dependent orthogonal matrix $\mathcal{R} = \mathcal{R}(t)$. This matrix describes the rotation of the body about the point O. The *angular velocity of the body* is defined to be the angular velocity vector $\boldsymbol{\omega}$ of $\mathcal{S}$ defined in (4.5). An important fact is that this is independent of the point O we picked. To see this note that any other point O' fixed in the body will have position vector $\mathbf{r}'$ relative to O, and the axes $\{\mathbf{e}'_i\}$ for the corresponding frame $\mathcal{S}'$ will be *fixed* relative to the axes $\{\mathbf{e}_i\}$. The

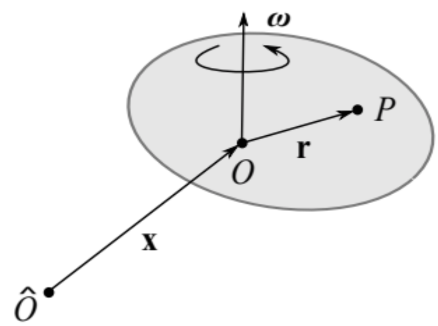

Figure 11: We fix a point O in the rigid body, which is taken to be the origin of the rest frame $\mathcal{S}$ of the body. The frame $\mathcal{S}$ has angular velocity $\boldsymbol{\omega}$, and its origin O has position vector $\mathbf{x}$ relative to the origin $\hat{O}$ of an inertial frame $\hat{\mathcal{S}}$. Any other point P fixed in the body has position vector $\mathbf{r}$ relative to O.

angular velocity $\boldsymbol{\omega}'$ of $\mathcal{S}'$ relative to $\hat{\mathcal{S}}$ is hence the same as the angular velocity $\boldsymbol{\omega}$ of $\mathcal{S}$ relative to $\hat{\mathcal{S}}$.

Since the rotation of the body is described entirely by the matrix $\mathcal{R}(t) \in SO(3)$, the configuration space of rotations is hence precisely the special orthogonal group $SO(3)$. A 3×3 matrix has 9 components and $\mathcal{R}\mathcal{R}^T = \mathbb{1}$ imposes 6 relations, so this rotational configuration space has dimension 3. We will introduce an explicit set of generalized angular coordinates for this space in section 4.5. Altogether the rigid body has 6 degrees of freedom: the 3 components of the position vector $\mathbf{x}$ of the point O, plus these three angles describing the orientation of the axes of $\mathcal{S}$.

If now $\mathbf{r}$ denotes the position vector of a point P fixed in the body, measured from O, then the velocity of this point as measured in the inertial frame $\hat{\mathcal{S}}$ is

$$\begin{aligned} \mathbf{v} &= \hat{\mathrm{D}}(\mathbf{r} + \mathbf{x}) \\ &= \boldsymbol{\omega} \wedge \mathbf{r} + \mathbf{v}_O \ . \end{aligned} \tag{4.18}$$

Here $\mathbf{v}_O = \hat{\mathrm{D}}\mathbf{x}$ is the velocity of the point O, and we have used the Coriolis formula (4.11).

Momentum and angular momentum

Let us suppose that the distribution of mass in the body is defined by a density $\rho(\mathbf{r})$, so that the mass δm in a small volume δV centred at $\mathbf{r}$ is $\delta m = \rho(\mathbf{r})\, \delta V$. Here $\mathbf{r}$ is measured from O. The total mass of the body is hence

$$M = \int_R \rho(\mathbf{r})\, \mathrm{d}V \ . \tag{4.19}$$

We will also sometimes want to treat two-dimensional bodies, such as a flat disc, or one-dimensional bodies such as a rigid rod. In this case one replaces ρ by a *surface density*, or *line density*, respectively, and integrates over the surface or curve, respectively.

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An important quantity for the rigid body is its *centre of mass* G. With respect to the origin O this has position vector

$$\mathbf{r}_G = \frac{1}{M}\int_R \mathbf{r}\,\rho(\mathbf{r})\,\mathrm{d}V . \tag{4.20}$$

This is the obvious generalization of (2.93) for a collection of point masses, and is an average position weighted by mass. We will henceforth generally choose the origin O of the rest frame $\mathcal{S}$ of the body to be at the centre of mass, so $O = G$ and $\mathbf{r}_G = \mathbf{0}$. This is a natural choice, and as we will see leads to a number of simplifications in subsequent formulae.

We may now define the *total momentum* $\mathbf{P}$ of the body relative to the inertial frame $\hat{\mathcal{S}}$ by similarly dividing it into small masses $\delta m = \rho(\mathbf{r})\,\delta V$ centred at $\mathbf{r}$. Each element has velocity $\mathbf{v}$ given by (4.18), again relative to $\hat{\mathcal{S}}$, meaning that

$$\mathbf{P} = \int_R \rho\,\mathbf{v}\,\mathrm{d}V = \int_R \rho(\mathbf{r})(\boldsymbol{\omega}\wedge\mathbf{r}+\mathbf{v}_O)\,\mathrm{d}V . \tag{4.21}$$

If we now choose $O = G$ then the first term in (4.21) is zero (it is $\boldsymbol{\omega}\wedge M\mathbf{r}_G = \mathbf{0}$) and we obtain

$$\mathbf{P} = \int_R \rho(\mathbf{r})\,\mathbf{v}_G\,\mathrm{d}V = M\mathbf{v}_G . \tag{4.22}$$

Thus the total momentum is as if the whole mass M was concentrated at the centre of mass G.

We may examine the *total angular momentum* $\mathbf{L}$ *about* O in a similar way. We have

$$\mathbf{L} = \int_R \mathbf{r}\wedge\rho\,\mathbf{v}\,\mathrm{d}V = \int_R \rho(\mathbf{r})\,\mathbf{r}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}+\mathbf{v}_O)\,\mathrm{d}V . \tag{4.23}$$

If $O = G$ then the second term is zero, and using the vector triple product we may write

$$\mathbf{L} = \int_R \rho(\mathbf{r})\,\mathbf{r}\wedge(\boldsymbol{\omega}\wedge\mathbf{r})\,\mathrm{d}V = \int_R \rho(\mathbf{r})\,[(\mathbf{r}\cdot\mathbf{r})\boldsymbol{\omega}-(\mathbf{r}\cdot\boldsymbol{\omega})\mathbf{r}]\,\mathrm{d}V . \tag{4.24}$$

In components of the basis $\{\mathbf{e}_i\}$ for $\mathcal{S}$ this reads

$$L_i = \sum_{j=1}^{3}\mathcal{I}_{ij}\,\omega_j , \tag{4.25}$$

where we have introduced the *inertia tensor* $\mathcal{I} = (\mathcal{I}_{ij})$ with components

$$\mathcal{I}_{ij} = \int_R \rho(\mathbf{r})\;[(\mathbf{r}\cdot\mathbf{r})\delta_{ij}-r_i r_j]\,\mathrm{d}V . \tag{4.26}$$

Notice that this is defined with respect to the rest frame $\mathcal{S}$ of the body, so is independent of time t, and is symmetric $\mathcal{I} = \mathcal{I}^T$. In general the angular momentum $\mathbf{L} = \mathcal{I}\,\boldsymbol{\omega}$ is not in the same direction as the angular velocity $\boldsymbol{\omega}$, which is what leads to some of the peculiar properties of rigid body motion. In Cartesian coordinates $\mathbf{r} = (x, y, z)$ we have

$$\mathcal{I} = \int_R \rho(\mathbf{r})\begin{pmatrix} y^2+z^2 & -xy & -zx \\ -xy & z^2+x^2 & -yz \\ -zx & -yz & x^2+y^2 \end{pmatrix}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z . \tag{4.27}$$

Note carefully the form of the terms in this matrix. The diagonal entries are called the *moments of inertia*, while the off-diagonal terms are the *products of inertia*. However, since $\mathcal{I}$ is a real symmetric matrix we may also diagonalize it via an orthogonal transformation. That is, there is a (constant) orthogonal matrix $\mathcal{P} \in O(3)$ such that $\mathcal{P}\mathcal{I}\mathcal{P}^T$ is diagonal. Thus in the new basis

$$\mathcal{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} . \qquad (4.28)$$

The eigenvalues I_i, $i = 1, 2, 3$, are called the *principal moments of inertia*. Notice that we have precisely used the freedom to rotate the rest frame $\mathcal{S}$ of the body by a constant orthogonal transformation, so that the axes $\mathbf{e}_i$ are the eigenvectors of the inertia tensor $\mathcal{I}$, called the *principal axes*, with corresponding eigenvalues I_i. These eigenvalues are non-negative: if $\boldsymbol{\alpha}$ is an eigenvector of $\mathcal{I}$ with eigenvalue I then

$$I|\boldsymbol{\alpha}|^2 = \boldsymbol{\alpha}^T\mathcal{I}\boldsymbol{\alpha} = \int_R \rho(\mathbf{r}) \left[|\mathbf{r}|^2|\boldsymbol{\alpha}|^2 - (\mathbf{r}\cdot\boldsymbol{\alpha})^2\right] \mathrm{d}V \geq 0, \qquad (4.29)$$

where we have used the Cauchy-Schwarz inequality.

The principal axes may often be identified by symmetry considerations. For example, an *axisymmetric body* by definition has rotational symmetry about a particular axis. The centre of mass must then lie on this axis, which is one of the principal axes. One can see this directly from the formula (4.27), where without loss of generality we take the axis of symmetry to be the $\mathbf{e}_3$ direction, and write the integrals in cylindrical polar coordinates (2.3). Thus $x = \varrho\cos\phi$, $y = \varrho\sin\phi$ and $\rho = \rho(\varrho, z)$. In particular one deduces that $I_1 = I_2$, and all off-diagonal products of inertia are zero. Let's now look at some more explicit examples:

Example (uniform rectangular cuboid): We will mainly focus on *uniform* distributions of mass, in which the density $\rho =$ constant. If we take the cuboid to have side lengths $2a$, $2b$, $2c$ and mass M, then $\rho = M/(8abc)$. The centre of mass is the origin of the cuboid, and we take Cartesian axes aligned with the edges. It is then straightforward to see that the products of inertia in this basis are zero; for example

$$\mathcal{I}_{12} = -\frac{M}{8abc}\int_{x=-a}^{a}\int_{y=-b}^{b}\int_{z=-c}^{c} xy\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0 . \qquad (4.30)$$

Thus the edge vectors of the cuboid are the principal axes. We next compute

$$\int_{x=-a}^{a}\int_{y=-b}^{b}\int_{z=-c}^{c} \rho\, x^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{M}{8abc}\left[\frac{1}{3}x^3\right]_{-a}^{a} 2b\cdot 2c = \frac{Ma^2}{3} . \qquad (4.31)$$

The integrals involving y^2 and z^2 are of course similar, and we deduce that $\mathcal{I}$ has principal moments of inertia $I_1 = \frac{1}{3}M(b^2+c^2)$, $I_2 = \frac{1}{3}M(c^2+a^2)$, $I_3 = \frac{1}{3}M(a^2+b^2)$.

Example (uniform disc): As a two-dimensional example, consider a uniform disc of radius r and mass M. Thus $\rho = M/(\pi r^2)$, and due to the rotational symmetry the centre of mass must be at the origin of the disc. If we take the $\mathbf{e}_3$ axis perpendicular to the disc then due to the axisymmetry we know already that $\mathcal{I}$ is diagonal in this basis with $I_1 = I_2$. We compute in polar coordinates $x = \varrho\cos\phi$, $y = \varrho\sin\phi$

$$I_1 \;=\; \int \rho\, y^2\, \mathrm{d}x\, \mathrm{d}y \;=\; \frac{M}{\pi r^2}\int_{\varrho=0}^{r}\int_{\phi=0}^{2\pi} \varrho^2 \sin^2\phi\, \varrho\, \mathrm{d}\varrho\, \mathrm{d}\phi \;=\; \frac{1}{4}Mr^2\,. \tag{4.32}$$

Notice here that the integrand is $\rho(x^2+y^2-x^2) = \rho\, y^2$, as the body is two-dimensional and lies in the plane $z = 0$. Since $I_1 = I_2$ (which is easy enough to check explicitly) we only have to compute

$$I_3 \;=\; \int \rho\,(x^2+y^2)\, \mathrm{d}x\, \mathrm{d}y \;=\; \frac{M}{\pi r^2}\cdot 2\pi \int_{\varrho=0}^{r} \varrho^2\, \varrho\, \mathrm{d}\varrho \;=\; \frac{1}{2}Mr^2\,. \tag{4.33}$$

The inertia matrix $\mathcal{I}$ defined by (4.26) in general depends on the choice of origin O. The formulae above were computed assuming this is at the centre of mass $O = G$. If we call this $\mathcal{I}^{(G)}$, then the following result allows us to easily compute the inertia tensor $\mathcal{I}^{(P)}$ about another other point P fixed in the body:

The parallel axes theorem: Denote the vector from G to P by $\mathbf{c} = (c_1, c_2, c_3)$. Then

$$\mathcal{I}_{ij}^{(P)} \;=\; \mathcal{I}_{ij}^{(G)} + M\left[(\mathbf{c}\cdot\mathbf{c})\delta_{ij} - c_i c_j\right]\,. \tag{4.34}$$

Calling this a theorem is perhaps a bit generous. The position vector $\mathbf{r}'$ of a point measured from P is related to the position vector $\mathbf{r}$ measured from G via $\mathbf{r}' = \mathbf{r} - \mathbf{c}$. In Cartesian coordinates we then simply compute

$$\mathcal{I}_{ij}^{(P)} \;\equiv\; \int_R \rho(\mathbf{r})\left[(\mathbf{r}'\cdot\mathbf{r}')\delta_{ij} - r_i' r_j'\right]\mathrm{d}V \;=\; \int_R \rho(\mathbf{r})\left[(\mathbf{r}-\mathbf{c})\cdot(\mathbf{r}-\mathbf{c})\delta_{ij} - (r_i - c_i)(r_j - c_j)\right]\mathrm{d}V \tag{4.35}$$

If one now multiplies out all the brackets and recalls that by the definition of centre of mass we have $\int_R \rho\,\mathbf{r}\,\mathrm{d}V = \mathbf{0}$, then one obtains (4.34).

Notice that we may rephrase this by saying that the inertia matrix in any rest frame of the body (with origin at P) is the sum of two terms: the inertia matrix in a frame with origin at the centre of mass G with axes parallel to the axes at P, plus the inertia matrix of a single particle of mass M at $\mathbf{c}$, measured about the centre of mass G.

Kinetic energy

Finally, let us compute the kinetic energy of the body in the inertial frame $\hat{\mathcal{S}}$. Following the arguments we used above for momentum we have

$$T \;=\; \int_R \frac{1}{2}\rho\,|\mathbf{v}|^2 \mathrm{d}V\,, \tag{4.36}$$

where $\mathbf{v}$ is given by (4.18). Thus

$$T \;=\; \frac{1}{2}\int_R \rho\left(|\mathbf{v}_O|^2 + 2\mathbf{v}_O\cdot(\boldsymbol{\omega}\wedge\mathbf{r}) + |\boldsymbol{\omega}\wedge\mathbf{r}|^2\right)\mathrm{d}V \;. \tag{4.37}$$

Taking $O = G$ as the centre of mass, the middle term is zero while the first term may be integrated to give

$$T \;=\; \frac{1}{2}M|\mathbf{v}_G|^2 + \frac{1}{2}\int_R \rho\,(\boldsymbol{\omega}\wedge\mathbf{r})\cdot(\boldsymbol{\omega}\wedge\mathbf{r})\,\mathrm{d}V \;. \tag{4.38}$$

The first term is the kinetic energy due to the motion of the centre of mass relative to $\hat{O}$, and again is as though all the mass was concentrated at the centre of mass. The second term is the *rotational kinetic energy* of the body. Using the vector identity

$$(\boldsymbol{\omega}\wedge\mathbf{r})\cdot(\boldsymbol{\omega}\wedge\mathbf{r}) \;=\; \boldsymbol{\omega}\cdot(\mathbf{r}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}))\;, \tag{4.39}$$

together with the formula (4.24) for the total angular momentum $\mathbf{L}$ about O we may write

$$T \;=\; \frac{1}{2}M|\mathbf{v}_G|^2 + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}\;. \tag{4.40}$$

Notice that the rotational kinetic energy may also be written in various ways as

$$T_{\mathrm{rot}} \;=\; \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} \;=\; \frac{1}{2}\sum_{i=1}^{3}\omega_i L_i \;=\; \frac{1}{2}\sum_{i,j=1}^{3}\mathcal{I}_{ij}\omega_i\omega_j \;=\; \frac{1}{2}\sum_{i=1}^{3} I_i\omega_i^2\;. \tag{4.41}$$