

Basic Mathematics
Lecture 13
Introduction to Probability
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Introduction to lecture 13

This lecture introduces you to probability theory. It discusses basic probability concepts.

Intended learning outcomes

At the end of this lecture you will be able to;

- (i) Define probability space and types of probabilities.
- (ii) Carry out operations involving basic probability theory.

References

These lecture notes should be complemented with relevant topics in (Antony & Robert, 2006; Kahenya, 2017; Upton & Cook, 2001).

Introduction to probability

Probability theory provides the mathematical rules for assigning probabilities to outcomes of random experiments.

Consider a coin. If it is spun once, you either get a head or a tail. The appearance of a head or tail is called an event. The two appearances i.e., head or tails are also referred to as outcomes. This implies that an event is a subset of all collection of all possible outcomes of an experiment.

The spin is called a **trial**. The fact that if you spin a coin it is just as likely to turn up heads as tails, we say the coin is **fair**.

Therefore, the **chance** or **probability** or the **likelihood** of a head is $\frac{1}{2}$. Since the coin is regarded to be **fair/unbiased**.

The probability of an event happening is always between 0 and 1. The probability of an event occurring i.e., it is certain is 1 e.g., the probability that June follows May, the probability that the sun will rise tomorrow.

The probability of event not occurring i.e., it is impossible is 0 e.g., the probability that you will grow wings at age 700 years, the probability that we shall meet yesterday etc.

Types of Probability Spaces

(a) Discrete Probability Space

Sample space S is the set of all possible outcomes of an experiment.

A sample space is said to be discrete if it is countable. Example of discrete probability spaces:

- (i) Tossing a coin once or number of times.
- (ii) Rolling a dice once or a number of times.

Suppose the sample space S consists of $n(S)$ possible outcomes and that each has an equal chance of happening. If the number of outcomes in each event is $n(E)$, then the probability that the event occur is given as;

$$p(E) = \frac{n(E)}{n(S)}$$

Example 1: What is the probability of getting an odd number when a dice is tossed once?

Solution: The set of odd numbers $A = \{1,3,5\}$

$$\Rightarrow p(A) = p(1) \text{ or } p(3) \text{ or } p(5) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

(b) Continuous Probability Space

A space is said to be continuous if it has uncountable number of elements. For example, space of any real number between 0 and 1 or space of any point (x, y) in the unit circle i.e.,

$$\text{The space } \Omega = \{(x, y): x^2 + y^2 \leq 1\}$$

In general, P is described by probability density function $f(x)$, a non-negative function that integrates to 1 i.e.

$$\begin{cases} f(x) \geq 0, x \in \mathbb{R} \\ \int f(x) dx = 1 \end{cases}$$

Remark: Do not confuse uncountable number of elements with countably infinite number of elements e.g., tossing a coin until you get a head.

The probability of X taking a value in the interval (a, b) is given by the corresponding area (Upton & Cook, 2001) i.e.

$$p(a < x < b) = \int_a^b f(x) dx$$

Example 1: The continuous random variable X has probability density function given by

$$f(x) = \begin{cases} \frac{1}{3}x & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

Find $p(X > 2)$

Solution: The probability is given as;

$$\int_2^3 \frac{1}{3}x dx = \left[\frac{x^2}{6} \right]_2^3 = \frac{3^2}{6} - \frac{2^2}{6} = \frac{5}{6} \Rightarrow p(X > 2) = \frac{5}{6}$$

Example 2: The continuous random variable X has pdf given by;

$$f(x) = \begin{cases} (kx^2 + 1) & 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

Determine the value of k and $p(X < 2)$

Solution: By definition $\int f(x)dx = 1$. Thus,

$$\begin{aligned} \int_1^3 (kx^2 + 1)dx &= \left[\frac{kx^3}{3} + x \right]_1^3 = \left(\frac{27k}{3} + 3 \right) - \left(\frac{k}{3} + 1 \right) = \frac{26}{3}k + 2 \\ \Rightarrow \frac{26}{3}k + 2 &= 1 \therefore k = -\frac{3}{26} \end{aligned}$$

$$p(X < 2) = \int_1^2 (kx^2 + 1)dx = \int_1^2 \left(-\frac{3x^2}{26} + 1 \right) dx = \left[-\frac{x^3}{26} + x \right]_1^2 = \left(-\frac{8}{26} + 2 \right) - \left(-\frac{1}{26} + 1 \right)$$

$$p(X < 2) = \frac{19}{26} \approx 0.7308$$

Properties or Rules of Probability

- a) If A is an event, then the probability of it occurring is $P(A) \leq 1$
- b) $P(\emptyset) = 0$
- c) If event A and event B are mutually exclusive, then $p(A \text{ or } B) = p(A) + p(B)$

Example 1: The ages of 120 college students in a certain class were recorded as below:

Ages	19	20	21	22	23	24	25
f	10	14	19	34	20	12	11

Determine the probability of selecting a student who is either 19 years or 22 years old.

Solution: $p(19 \text{ or } 22) = p(19) + p(22) = \frac{10}{120} + \frac{34}{120} = \frac{44}{120} = \frac{11}{30}$

d) Complementation Rule

For any event A, $P(A) = 1 - P(\text{not } A)$ i.e., the probability that an event A occurs equals one minus the probability that its complement occurs.

Example 1: The ages of 120 college students in a certain class were recorded as below:

Ages	19	20	21	22	23	24	25
f	10	14	19	34	20	12	11

Determine the probability of selecting a student who is less than 24 years old.

Solution: Let A be the event that student is less than 24 yrs.

By the complementation rule $P(A) = 1 - P(A')$ where A' is the event that student is 24 plus years old.

$$P(A) = 1 - P(A') = 1 - \frac{23}{120} = \frac{97}{120}$$

e) Joint Probability

If A and B are two independent events i.e., the occurrence of A does not influence the occurrence of B and vice versa, then

$$p(A \text{ and } B) = p(A) \times p(B)$$

Example 1: A fair coin is tossed twice. Find the probability of getting a head in the first toss and in the second toss.

Solution: Note that the outcome of the second toss is independent of the outcome of the first toss.

$$p(H \text{ and } H) = p(H) \times p(H) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Example 2: One can also use the contingency tables to work out the joint probability. For instance, the table below represents the possible outcomes of tossing a fair coin (C) once and rolling a fair dice (D) once.

C \ D	1	2	3	4	5	6
H	H1	H2	H3	H4	H5	H6
T	T1	T2	T3	T4	T5	T6

Find the probability of getting a 3 and a head.

Solution: $p(3 \text{ and } H) = p(3) \times p(H) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$

f) Conditional probability

If A and B are events, then the probability that event B occurs given that event A has occurred is called a conditional probability, denoted $P(B|A)$ (read as, ‘Probability of B given A’)

Example 1: A die is tossed. What is the probability of getting a 2 given that the die comes up even?

Solution: Let A = getting a 2; Let B = getting an even number, then

$$P(A|B) = \frac{1}{3}$$

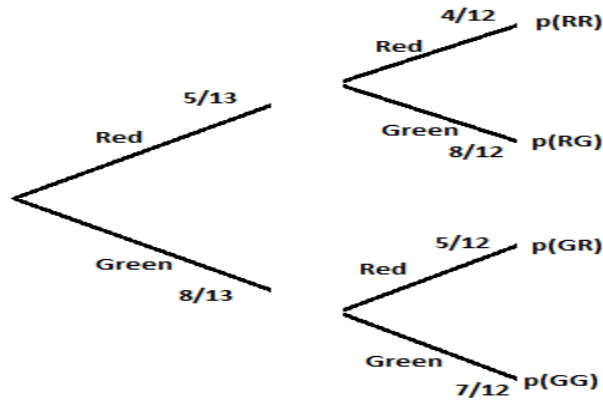
Use of a tree diagram

Use of tree diagram is a convenient way of calculating the probabilities especially when dealing with the multiplication rule i.e., if A and B are any two events, then

$P(A \text{ and } B) = P(A)P(B|A)$. That is the probability that both events A and B occur equals the probability that event A occurs times the conditional probability that event B occurs given that event A occurs.

Example 1: Two marbles are selected at random from a bowl containing 5 red marbles and 8 green marbles. What is the probability of picking a red marble and a green marble in that order, without replacement?

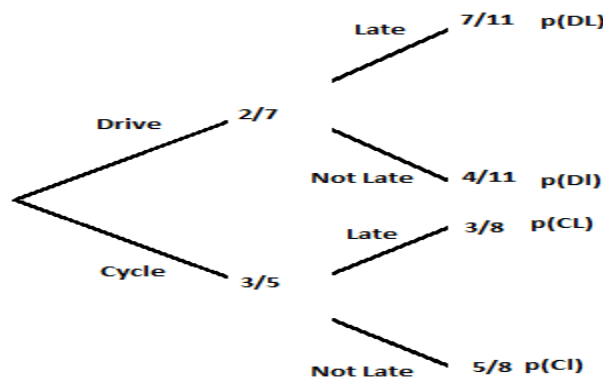
Solution: Using a tree diagram



$$p(\text{Red and Green}) = p(R) \times p(G) = p(RG) = \frac{5}{13} \times \frac{8}{12} = \frac{40}{156} = \frac{10}{39}$$

Example 2: Onyango can drive or cycle to work. The probability that he will drive in a certain day is $\frac{2}{7}$ and that he will cycle to work is $\frac{3}{5}$. If Onyango drive to work, the probability of being late for work is $\frac{7}{11}$ and if he cycles and is late for work is $\frac{3}{8}$. Find the probability that he will not be late for work.

Solution:



$$\text{The } p(\text{not late for work}) = p(DI) \text{ or } p(CI) = \left(\frac{2}{7} \cdot \frac{4}{11}\right) + \left(\frac{3}{5} \cdot \frac{5}{8}\right) = \frac{8}{77} + \frac{3}{8} = \frac{295}{616}$$

Exercise

1. A fair coin and a dice are tossed at the same time. Determine the probability of getting a tail and a 3.
2. A fair coin and a dice are tossed at the same time. Determine the probability of getting a head and an even number.
3. Two dice are tossed at the same time. Draw the contingency table for the total possible outcomes.
4. What is the probability that a natural number chosen at random from the first 40 natural numbers is a multiple of 36?
5. A bag contains 9 black marbles and 11 red marbles. If two marbles are drawn from the bag, one at a time, find the probability of drawing a black and a red marble without replacement and with replacement.
6. Given that pdf, $f(x) = \begin{cases} x^2 - 2 & 0 < x < 3 \\ 0 & \text{Otherwise} \end{cases}$. Determine (i) $p(X > 1)$ (ii) $p(X < 2)$
7. Given that pdf, $f(x) = \begin{cases} \frac{1}{3}x^2 & 0 < x < 3 \\ 0 & \text{Otherwise} \end{cases}$. Determine (i) $p(X > 1)$ (ii) $p(X < 2)$
8. Given that pdf, $f(x) = \begin{cases} kx^2 + \frac{1}{3} & 0 < x < 3 \\ 0 & \text{Otherwise} \end{cases}$. Determine the value of k and $p(X < 1)$
9. Given that pdf, $f(x) = \begin{cases} kx^2 & -3 < x < 3 \\ 0 & \text{Otherwise} \end{cases}$. Determine the value of k , $p(X < 1)$, $p(-1 < X < 1)$
10. Two similar bowls A and B contains red and green marbles. Bowl A contains 13 red marbles and 17 green marbles, while bowl B contains 7 red marbles and 10 green marbles. A bowl is randomly pick and two marbles are randomly pick, one at a time and without replacement. Find the probability of;
 - (i) Picking two reds
 - (ii) Picking a red and a green
 - (iii) Picking greens.

Bibliography

Antony, C., & Robert, D. (2006). *Foundation Maths*. Prentice Hall.

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Upton, G., & Cook, I. (2001). *Introduction to Statistics* (2nd ed.). Oxford University Press.