OPERATIONS RESEARCH

LECTURE ONE

Review of Matrices

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MATRICES

A matrix is a rectangular array of items or numbers known as elements. These elements are arranged in rows and columns to represent some information.

The position of an element in one matrix is very important and is located by the number of the row and column which it occupies.

The size of a matrix is defined by the number of its rows (m) and column (n) thus, $m \times n$ read as m by n.

For example, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2\times 2}$ and $B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}_{3\times 3}$

Matrix A has 2 rows and 2 columns and B has 3 rows and 3 columns.

A matrix A with three rows and four columns is generalized as:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \text{ or } A = (a_{ij})\ i[rows] = 1,2,3;\ \ j[columns] = 1,2,3,4$$

Properties of matrices

Equal Matrices - Two matrices A and B are said to be equal, that is $A = B$ or $(a_{ij}) = (b_{ij})$ if and only if they have the same order and the elements in the corresponding locations in the two matrices are the same, that is, $a_{ij} = b_{ij}$ for all i *and* j.

Example

The following matrices are equal $\begin{pmatrix} 5 & 6 & 7 \\ 0 & 3 & 9 \\ 2 & 8 & 0 \\ 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 6 & 7 \\ 0 & 3 & 9 \\ 2 & 8 & 0 \\ 1 & 4 & 2 \end{pmatrix}$

Column Matrix or vector

A column matrix, also referred to as column vector is a matrix consisting of a single column.

For example, $A = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}_{n\times 1}$

Row matrix or vector

This is a matrix with a single row

For example, $A = (a_{11} \quad a_{12} \quad \cdots \quad a_{1n})_{1\times n}$

Transpose of a Matrix

The transpose of matrix A denoted as A^T, is obtained by interchanging the rows and columns of the given matrix. If A is an $m \times n$ matrix then A^T will be a $n \times m$ matrix. That is:

If $A = (a_{ij})_{m\times n}$, then $A^T = (a_{ji})_{n\times m}$

Examples

Find the transposes of the following matrices

a. $A = \begin{pmatrix} 1 & 4 & 8 \\ 0 & 3 & 5 \end{pmatrix}$

b. $B = \begin{pmatrix} 9 & 7 \\ 5 & 3 \end{pmatrix}$

c. $C = (x_1 \quad x_2 \quad x_3 \quad x_4)$

Solution

a. $A^T = \begin{pmatrix} 1 & 4 & 8 \\ 0 & 3 & 5 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 4 & 3 \\ 8 & 5 \end{pmatrix}$

b. $B^T = \begin{pmatrix} 9 & 7 \\ 5 & 3 \end{pmatrix}^T = \begin{pmatrix} 9 & 5 \\ 7 & 3 \end{pmatrix}$

c. $C^T = (x_1 \quad x_2 \quad x_3 \quad x_4)^T = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

Square Matrix

A matrix is said to be square when its number of rows and columns are equal.

i.e., $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_{n \times n}$

Diagonal matrices

It is a square matrix whose elements are all zeros except at least one element on the principal diagonal

e.g. $A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}$, $\qquad B = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

An identity or unity matrix

It is a diagonal matrix in which each of the diagonal elements is a positive one (1) e.g., $I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ $n \times n$ unit matrix

A null or zero matrix

A null or zero matrix is a matrix whose elements are all zeros.

Sub matrix

This is a matrix obtained by deleting selected row(s) and/or column(s) of a given matrix.

e.g., if $A = \begin{pmatrix} 2 & 0 & 7 \\ 4 & 6 & 9 \\ 3 & 1 & 0 \end{pmatrix}$, then $A_1 = \begin{pmatrix} 2 & 0 & 7 \\ 4 & 6 & 9 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 4 & 9 \\ 3 & 0 \end{pmatrix}$ are some sub matrices of matrix A.

OPERATION ON MATRICES

Matrix addition and subtraction

Two or more matrices are compatible in addition or subtraction if they are of the same order. We add or subtract elements in corresponding positions.

Example:

Given matrix A and B, calculate $A + B$ and $A - B$

$$A = \begin{pmatrix} 1 & -4 & 9 \\ 0 & 16 & 1 \\ -2 & 4 & 3 \\ 7 & 8 & 15 \end{pmatrix} \quad B = \begin{pmatrix} 14 & 7 & 9 \\ -3 & 0 & 11 \\ 5 & 6 & -7 \\ 2 & -14 & 8 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 1 & -4 & 9 \\ 0 & 16 & 1 \\ -2 & 4 & 3 \\ 7 & 8 & 15 \end{pmatrix} + \begin{pmatrix} 14 & 7 & 9 \\ -3 & 0 & 11 \\ 5 & 6 & -7 \\ 2 & -14 & 8 \end{pmatrix} = \begin{pmatrix} 15 & 3 & 18 \\ -3 & 16 & 12 \\ 3 & 10 & -4 \\ 9 & -6 & 23 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 1 & -4 & 9 \\ 0 & 16 & 1 \\ -2 & 4 & 3 \\ 7 & 8 & 15 \end{pmatrix} - \begin{pmatrix} 14 & 7 & 9 \\ -3 & 0 & 11 \\ 5 & 6 & -7 \\ 2 & -14 & 8 \end{pmatrix} = \begin{pmatrix} -13 & -11 & 0 \\ 3 & 16 & -10 \\ -7 & -2 & 10 \\ 5 & 22 & 7 \end{pmatrix}$$

If it is assumed that A, B, C are of the same order, the following properties are fulfilled:

a) Commutative law: A + B = B + A

b) Associative law: (A + B) + C = A + (B + C) = A + B + C

Multiplying a matrix by a scalar

In this case each element of the matrix is multiplied by that scalar

Example

$$\text{If } A = \begin{pmatrix} 6 & -1 & 10 & 5 \\ 3 & 4 & 2 & -5 \\ -9 & 13 & -6 & 0 \end{pmatrix}$$

$$\text{then } (10)A = \begin{pmatrix} 60 & -10 & 100 & 50 \\ 30 & 40 & 20 & -50 \\ -90 & 130 & -60 & 0 \end{pmatrix}$$

Multiplication of two matrices

Two matrices are compatible in multiplication if the number of columns in the first matrix equals the number of rows in the second matrix. The order on the product then becomes the number of rows of the first matrix by the number of columns of the second matrix. This means that if A is a matrix of order $m \times n$, it can only be compatible in multiplication to a matrix B whose order must be $n \times p$ such that their product matrix C, i.e., $A \times B = C$, will be of order $m \times p$.

Generally, we multiply the the elements in row m of the first matrix by the corresponding elements in columns p of the second matrix and the products obtained are then added giving a single number.

We can express this rule as follows

Given $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}_{3\times 2}$ and $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}_{2\times 3}$, then

$$A \times B = C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{3\times 3}$$

Where $c_{11} = (a_{11} \times b_{11}) + (a_{12} \times b_{12})$ and so on…

Example I

Evaluate; $\begin{bmatrix} 2 & 4 & 2 \\ 3 & 2 & 4 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & 0 \\ 3 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 18 & 10 & 4 \\ 16 & 16 & 2 \\ 12 & 5 & 3 \end{bmatrix}$

Example II

Given that matrix $A = \begin{bmatrix} -1 & -6 \\ 0 & 8 \end{bmatrix}$ $and\ B = \begin{bmatrix} 2 & -3 \\ 5 & -7 \end{bmatrix}$, Find $A^T + AB + 2(A - B)$

Solution:

$$A^T = \begin{pmatrix} 1 & 0 \\ -6 & 8 \end{pmatrix}$$

$$AB = \begin{pmatrix} -1 & -6 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 5 & -7 \end{pmatrix} = \begin{pmatrix} -32 & 45 \\ 40 & -56 \end{pmatrix}$$

$$2(A - B) = 2\begin{pmatrix} -3 & -3 \\ -5 & 15 \end{pmatrix} = \begin{pmatrix} -6 & -6 \\ -10 & 30 \end{pmatrix}$$

$$A^T + AB + 2(A - B) = \begin{pmatrix} 1 & 0 \\ -6 & 8 \end{pmatrix} + \begin{pmatrix} -32 & 45 \\ 40 & -56 \end{pmatrix} + \begin{pmatrix} -6 & -6 \\ -10 & 30 \end{pmatrix}$$

$$= \begin{pmatrix} -37 & 39 \\ 24 & 18 \end{pmatrix}$$

<u>**Exercise**</u>

Give matrix $A = \begin{pmatrix} 3 & 1 & -1 \\ 4 & 1 & -2 \\ -5 & -2 & 3 \end{pmatrix}$, determine A^2 and $A^T - \frac{1}{2}A$

The determinant of a square matrix

The determinant of square matrix A is denoted as $\det(A)$ or $|A|$ is a number associated to that matrix. A matrix whose determinant is equal to zero, is known as a singular matrix otherwise it is a non-singular matrix.

i. Determinant of a 2×2 matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(A)$ or $|A| = ad - bc$

ii. Determinant of a 3×3 matrix

Given $B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, then $\det(B)$ or $|B| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Inverse of a matrix

If A is a non-singular matrix and there exists another non-singular matrix denoted by A^{-1} such that the product of the two is an identity matrix I_n, that is $A \times A^{-1} = A^{-1} \times A = I_n$, then A^{-1} is said to be inverse of A and vice versa.