

# OPERATIONS RESEARCH

## LECTURE TWO

### Input - Output Models (1)

**Lecturer: Dr. Emily Roche**

#### INTRODUCTION

This lecture will focus on review of the introduction to Input - Output models.

#### Intended learning outcomes

At the end of this lecture, you will be able to:

- i. Explain properties of Input-output problem
- ii. Formulate the Technical coefficients matrix

#### References

These lecture notes should be supplemented with relevant topics from the books listed in the Bibliography at the end of the lecture

#### INPUT - OUTPUT ANALYSIS

The input output analysis is a topic which requires application of matrices

The technique analyses the flow of inputs from one sector of the economy to the other sectors thus the technique is quite useful in studying the interdependence of sectors within a single economy.

#### The economic system

This is an organized way in which a state or nation allocates its resources and apportions goods and services to the national community.

It is used to control the factors of production such as labour, capital, entrepreneurs, physical resources, and intellectual resources.

This system is studied using Input-Output analysis which is the technique used in economics for resources and products within an economy.

## Input-Output matrix

These show the units required from each industry/sector to make up one complete product in each industry/sector. Suppose an economic system has  $n$  industries  $I_1, I_2, I_3, \dots, I_n$  In each of which has inputs (raw materials/utilities) and output (finished products). The input coefficient ( $d_{ij}$ ) measures the amount of input the  $j^{th}$  industry needs from the  $i^{th}$  industry to produce 1 complete unit.

The collection of input coefficients is given by the following matrix:

$$D = \begin{matrix} \begin{pmatrix} d_{11} & d_{12} & \dots & d_{1n} \\ d_{21} & d_{22} & \dots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \dots & d_{nn} \end{pmatrix} & \begin{matrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{matrix} \\ \begin{matrix} I_1 & I_2 & \dots & I_n \end{matrix} & \begin{matrix} Suppliers \\ \\ \\ Users \end{matrix} \end{matrix}$$

The matrix  $D$  is called the input-output matrix/technology matrix/Leontief input - output matrix.

The input - output analysis was first developed by Prof Leontief hence the Leontief matrix.

### Illustration:

Suppose the entries of  $D$  are given in dollars for instance,  $d_{12} = \$0.43$ , then \$0.43 worth of Industry one's product must be used to produce \$1 worth of Industry two's product.

The total amount spent by the  $j^{th}$  industry to produce \$1 worth of output is given by the sum of the entries in the  $j^{th}$  column.

A matrix will qualify as an Input-Output matrix if it has the following characteristics:

1. It MUST be a square matrix
2. Each element of the matrix must be in the range  $[0,1]$  i.e.,  $0 \leq d_{ij} \leq 1$
3. The sum of elements in any given column must be less than or equal to ones  $S \leq 1$ .

### Example

A system composed of two industries the Coal and Steel industry has the following inputs to produce \$1 worth of output. The Coal industry requires \$0.1 of its product and \$0.8 of Steel. The Steel industry requires \$0.1 of its own and \$0.2 of Coal. Form the input-output matrix.

### Solution

$$D = \begin{array}{cc} \begin{array}{cc} \text{Coal} & \text{Steel} \\ \begin{pmatrix} 0.1 & 0.2 \\ 0.8 & 0.1 \end{pmatrix} & \begin{array}{c} \text{Coal} \\ \text{Steel} \end{array} \\ \text{Users} & \text{Suppliers} \end{array} & \therefore D = \begin{pmatrix} 0.1 & 0.2 \\ 0.8 & 0.1 \end{pmatrix} \end{array}$$

### Example

A system is composed of three industries with the following inputs:

P requires \$0.1 of its own, \$0.15 of Q and \$0.23 of R

Q requires \$0.43 of its P, \$0.0 of its own and \$0.03 of R

R requires \$0.0 of P, \$0.37 of Q and \$0.02 of its own.

Form the input-output matrix

### Solution

$$D = \begin{array}{ccc} \begin{pmatrix} 0.1 & 0.43 & 0.0 \\ 0.15 & 0.0 & 0.37 \\ 0.23 & 0.03 & 0.02 \end{pmatrix} & \begin{array}{c} P \\ Q \\ R \end{array} \end{array}$$

### Open and Closed economic system

From the developed input-output model, we can denote the total output of the  $i^{th}$  industry by  $X_i$ . If the economic system is closed, then it means that it sells the products only to industries within the system. The total output of the  $i^{th}$  industry is given by:

$$X_i = d_{i1}X_1 + d_{i2}X_2 + d_{i3}X_3 + \dots + d_{in}X_n$$

Thus, the entire system of outputs for a closed model is:

$$\begin{aligned}
 X_1 &= d_{11}X_1 + d_{12}X_2 + d_{13}X_3 + \dots + d_{1n}X_n \\
 X_2 &= d_{21}X_1 + d_{22}X_2 + d_{23}X_3 + \dots + d_{2n}X_n \\
 X_3 &= d_{31}X_1 + d_{32}X_2 + d_{33}X_3 + \dots + d_{3n}X_n \\
 &\dots \\
 X_n &= d_{n1}X_1 + d_{n2}X_2 + d_{n3}X_3 + \dots + d_{nn}X_n
 \end{aligned}$$

In matrix form  $X = DX$

$$\text{Where } X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} d_{11} & d_{12} & d_{13} & \dots & d_{1n} \\ d_{21} & d_{22} & d_{23} & \dots & d_{2n} \\ d_{31} & d_{32} & d_{33} & \dots & d_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & d_{3n} & \dots & d_{nn} \end{pmatrix}$$

For a closed system, the sum of the elements of each column must be 1.

A more common application of the Leontief input-output matrix model occurs when the industries within the system sells products to those outside the system. Such systems are called open systems and the total output of the  $i^{th}$  industry is given by:

$$X_i = d_{i1}X_1 + d_{i2}X_2 + d_{i3}X_3 + \dots + d_{in}X_n + e_i$$

Where  $e_i$  represents the external demand for the  $i^{th}$  industry.

The collection of the output is given by the linear system:

$$\begin{aligned}
 X_1 &= d_{11}X_1 + d_{12}X_2 + d_{13}X_3 + \dots + d_{1n}X_n + e_1 \\
 X_2 &= d_{21}X_1 + d_{22}X_2 + d_{23}X_3 + \dots + d_{2n}X_n + e_2 \\
 X_3 &= d_{31}X_1 + d_{32}X_2 + d_{33}X_3 + \dots + d_{3n}X_n + e_3 \\
 &\dots \\
 X_n &= d_{n1}X_1 + d_{n2}X_2 + d_{n3}X_3 + \dots + d_{nn}X_n + e_n
 \end{aligned}$$

In matrix form  $X = DX + E$

$$\begin{aligned} \Rightarrow X - DX &= E \\ X(I - D) &= E \\ (I - D)^{-1}(I - D)X &= (I - D)^{-1}E \\ X &= (I - D)^{-1}E \end{aligned}$$

Where  $(I - D)^{-1}$  is known as the Leontief inverse matrix and

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{pmatrix}; \quad D = \begin{pmatrix} d_{11} & d_{12} & d_{13} & \dots & d_{1n} \\ d_{21} & d_{22} & d_{23} & \dots & d_{2n} \\ d_{31} & d_{32} & d_{33} & \dots & d_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & d_{n3} & \dots & d_{nn} \end{pmatrix}; \quad E = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_n \end{pmatrix}; \quad I = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

An open economic system has the sum of elements of each column being less than one.

### Determining the Inverse of a square matrix

For any non-singular square matrix  $A$ , there exists another non-singular square matrix of the same dimension  $A^{-1}$  such that  $AA^{-1} = A^{-1}A = I_n$ . The matrices  $A$  and  $A^{-1}$  are inverses of each other.

### Finding inverse of a $2 \times 2$ matrix

The first step to finding the inverse of a matrix is to verify that the matrix is non-singular. This is then followed by computing the inverse.

$$\text{Let } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ then } |A| = ad - bc$$

$$\text{Subsequently, } A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

### Finding inverse of a $3 \times 3$ matrix

$$\text{Let } B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \text{ using Sarrus method } |B| = (aei + bfg + cdh) - (bdi + afh + ceg)$$

To enable us to find the inverse of a  $3 \times 3$  matrix, we need to learn about the minor, cofactor and adjoint of a  $3 \times 3$  matrix.

### The matrix of minors ( $M$ ).

This is a matrix whose elements are the determinants of all the  $2 \times 2$  submatrices obtained from the matrix.

Let matrix  $A = (a_{ij})$ , the minor element  $m_{ij}$  of the element  $a_{ij}$  is the determinant of the submatrix obtained by deleting row  $i$  and column  $j$  in matrix  $A$ .

### The cofactor matrix ( $C$ )

This is a matrix whose elements are 'signed' elements of the matrix of minors. The signs to be assigned to each element alternate both vertically and horizontally starting with a  $+$  for element  $m_{11}$ .

For instance, the signs to be attached to the elements of the matrix of minors for a

$3 \times 3$  matrix are  $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

### The adjoint matrix ( $C^T$ )

This is the transpose of the cofactor matrix

Therefore the inverse of the matrix will be given as  $\frac{1}{\text{determinant}} \times C^T$

### Example

Determine the inverse of  $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$

### Solution

$$|A^{-1}| = (0 + 0 + 24) - (4 + 3 + 0) = 17$$

This confirms that matrix  $A$  is non-singular

The elements of the minors matrix are:

$$\begin{aligned}m_{11} &= \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = -3 & m_{12} &= \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2 & m_{13} &= \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6 \\m_{21} &= \begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} = -10 & m_{22} &= \begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} = 1 & m_{23} &= \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 3 \\m_{31} &= \begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} = 2 & m_{32} &= \begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} = -7 & m_{33} &= \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = -4\end{aligned}$$

$$\therefore \text{Matrix of minors, } M = \begin{pmatrix} -3 & 2 & 6 \\ -10 & 1 & 3 \\ 2 & -7 & -4 \end{pmatrix}$$

$$\text{Cofactor matrix, } C = \begin{pmatrix} -3 & -2 & 6 \\ 10 & 1 & -3 \\ 2 & 7 & -4 \end{pmatrix}$$

$$\text{Adjoint matrix, } C^T = \begin{pmatrix} -3 & 10 & 2 \\ -2 & 1 & 7 \\ 6 & -3 & -4 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{17} \begin{pmatrix} -3 & 10 & 2 \\ -2 & 1 & 7 \\ 6 & -3 & -4 \end{pmatrix}.$$

You can then verify that  $AA^{-1} = I_3$

### **Bibliography**

Lucey, T. (2002). *Quantitative Techniques* (6th ed.). Cengage Learning.

Taha, H. A. (2017). *Operation Research An introduction* (10th ed.). Prentice-Hall, Inc.