

OPERATIONS RESEARCH

LECTURE THREE

Input - Output Models (2)

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INTRODUCTION

This lecture will focus on solving problems involving input-output analysis

Intended learning outcomes

At the end of this lecture, you will be able to solve two and three input-output industry models

References

These examples should be supplemented with relevant workings from the books listed in the Bibliography at the end of the lecture.

Example 1

A system has two industries with the following inputs:

Industry *A* requires 0.3 of its own and 0.4 of *B*'s product, while *B* requires 0.2 of its own and 0.4 of *A*'s.

- i. Form the input-output matrix
- ii. Obtain the total output of every industry given that the external demand for *A* is 50,000 and for *B* is 30,000.

Solution

i.
$$D = \begin{matrix} & \begin{matrix} A & B \end{matrix} \\ \begin{pmatrix} 0.3 & 0.4 \\ 0.4 & 0.2 \end{pmatrix} & \begin{matrix} A \\ B \end{matrix} \end{matrix} \quad \begin{matrix} \text{Suppliers} \\ \text{Users} \end{matrix}$$

ii. $X = (I - D)^{-1}E$

$$I - D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.3 & 0.4 \\ 0.4 & 0.2 \end{pmatrix} = \begin{pmatrix} 0.7 & -0.4 \\ -0.4 & 0.8 \end{pmatrix}$$

$$|I - D| = (0.7 \times 0.8) - (-0.4 \times -0.4) = 0.4$$

$$(I - D)^{-1} = \frac{1}{0.4} \begin{pmatrix} 0.8 & 0.4 \\ 0.4 & 0.7 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1.75 \end{pmatrix}$$

$$X = \begin{pmatrix} 2 & 1 \\ 1 & 1.75 \end{pmatrix} \begin{pmatrix} 50000 \\ 30000 \end{pmatrix} = \begin{pmatrix} 130,000 \\ 102,500 \end{pmatrix}$$

$$X_A = 130,000$$

$$X_B = 102,500$$

Example 2

There are three industries in a system namely transportation, Coal mining and Electricity with the following input matrix D and external demand vector E

$$D = \begin{pmatrix} 0.15 & 0.03 & 0.15 \\ 0.50 & 0.0 & 0.40 \\ 0.20 & 0.10 & 0.20 \end{pmatrix} \quad E = \begin{pmatrix} 450 \\ 955 \\ 670 \end{pmatrix}$$

Find the output for each industry

Solution

$$X = (I - D)^{-1}E$$

$$(I - D) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.15 & 0.03 & 0.15 \\ 0.50 & 0.0 & 0.40 \\ 0.20 & 0.10 & 0.20 \end{pmatrix} = \begin{pmatrix} 0.85 & -0.03 & -0.15 \\ -0.50 & 1 & -0.40 \\ -0.20 & -0.10 & 0.80 \end{pmatrix}$$

$$\begin{aligned} |I - D| &= 0.85 \begin{vmatrix} 1 & -0.40 \\ -0.10 & 0.80 \end{vmatrix} + 0.03 \begin{vmatrix} -0.50 & -0.40 \\ -0.20 & 0.80 \end{vmatrix} - 0.15 \begin{vmatrix} -0.50 & 1 \\ -0.20 & -0.10 \end{vmatrix} \\ &= 0.646 - 0.144 - 0.0375 = 0.4645 \end{aligned}$$

The matrix of minors M is obtained as:

$$M = \begin{pmatrix} 0.76 & -0.48 & 0.25 \\ -0.255 & 0.65 & -0.145 \\ 0.27 & -0.415 & 0.70 \end{pmatrix}$$

The cofactor matrix then becomes:

$$C = \begin{pmatrix} 0.76 & 0.48 & 0.25 \\ 0.255 & 0.65 & 0.145 \\ 0.27 & 0.415 & 0.70 \end{pmatrix}$$

The adjoint matrix is:

$$C^T = \begin{pmatrix} 0.76 & 0.255 & 0.27 \\ 0.48 & 0.65 & 0.415 \\ 0.25 & 0.145 & 0.70 \end{pmatrix}$$

$$\therefore X = (I - D)^{-1}E = \frac{1}{0.4645} \begin{pmatrix} 0.76 & 0.255 & 0.27 \\ 0.48 & 0.65 & 0.415 \\ 0.25 & 0.145 & 0.70 \end{pmatrix} \begin{pmatrix} 450 \\ 955 \\ 670 \end{pmatrix} = \begin{pmatrix} 1650 \\ 2400 \\ 1550 \end{pmatrix}$$

Example 3

Find the output matrix for an economic system with input matrix D and E given by:

$$D = \begin{pmatrix} 0.1 & 0.43 & 0.00 \\ 0.15 & 0.00 & 0.37 \\ 0.23 & 0.03 & 0.02 \end{pmatrix} \quad E = \begin{pmatrix} 20,000 \\ 30,000 \\ 30,000 \end{pmatrix}$$

Solution

$$X = (I - D)^{-1}E$$

$$(I - D) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.1 & 0.43 & 0.00 \\ 0.15 & 0.00 & 0.37 \\ 0.23 & 0.03 & 0.02 \end{pmatrix} = \begin{pmatrix} 0.90 & -0.43 & 0.00 \\ -0.15 & 1 & -0.37 \\ -0.23 & -0.03 & 0.98 \end{pmatrix}$$

$$\begin{aligned} |I - D| &= 0.90(0.9689) + 0.43(-0.2321) + 0.00(0.2345) \\ &= 0.772207 \end{aligned}$$

The matrix of minors M is obtained as:

$$M = \begin{pmatrix} 0.9689 & -0.2321 & 0.2345 \\ -0.4214 & 0.882 & -0.1259 \\ 0.1591 & -0.333 & 0.8355 \end{pmatrix}$$

The cofactor matrix then becomes:

$$C = \begin{pmatrix} 0.9689 & 0.2321 & 0.2345 \\ 0.4214 & 0.882 & 0.1259 \\ 0.1591 & 0.333 & 0.8355 \end{pmatrix}$$

The adjoint matrix is:

$$C^T = \begin{pmatrix} 0.9689 & 0.4214 & 0.1591 \\ 0.2321 & 0.882 & 0.333 \\ 0.2345 & 0.1259 & 0.8355 \end{pmatrix}$$

$$\begin{aligned} \therefore X &= (I - D)^{-1}E = \frac{1}{0.772207} \begin{pmatrix} 0.9689 & 0.4214 & 0.1591 \\ 0.2321 & 0.882 & 0.333 \\ 0.2345 & 0.1259 & 0.8355 \end{pmatrix} \begin{pmatrix} 20,000 \\ 30,000 \\ 30,000 \end{pmatrix} \\ &= \begin{pmatrix} 47,646.55 \\ 53,213.71 \\ 43,423.59 \end{pmatrix} \end{aligned}$$

Example 4

An economic system is composed of three industries A, B and C . Given that industry C 's production is 1230, find the production of the other two industries if

$$D = \begin{pmatrix} 0.3 & 0.45 & 0.2 \\ 0.25 & 0.4 & 0.5 \\ 0.45 & 0.15 & 0.3 \end{pmatrix}$$

Solution

The input-output matrix D , represents a closed economic system since the sum of elements in each column of D is equal to one.

So, $X = DX$

$$\begin{pmatrix} X_A \\ X_B \\ X_C \end{pmatrix} = \begin{pmatrix} 0.3 & 0.45 & 0.2 \\ 0.25 & 0.4 & 0.5 \\ 0.45 & 0.15 & 0.3 \end{pmatrix} \begin{pmatrix} X_A \\ X_B \\ X_C \end{pmatrix}$$

$$\begin{pmatrix} X_A \\ X_B \\ X_C \end{pmatrix} = \begin{pmatrix} 0.3X_A + 0.45X_B + 0.2X_C \\ 0.25X_A + 0.4X_B + 0.5X_C \\ 0.45X_A + 0.15X_B + 0.3X_C \end{pmatrix}$$

$$\begin{aligned} X_A &= 0.3X_A + 0.45X_B + 0.2X_C \\ X_B &= 0.25X_A + 0.4X_B + 0.5X_C \\ X_C &= 0.45X_A + 0.15X_B + 0.3X_C \end{aligned}$$

$$\begin{aligned} 0.7X_A - 0.45X_B - 0.2X_C &= 0 \\ -0.25X_A + 0.6X_B - 0.5X_C &= 0 \\ -0.45X_A - 0.15X_B + 0.7X_C &= 0 \end{aligned}$$

$$\begin{aligned} 70X_A - 45X_B - 20X_C &= 0 \\ -25X_A + 60X_B - 50X_C &= 0 \\ -45X_A - 15X_B + 70X_C &= 0 \end{aligned}$$

$$\begin{aligned} 14X_A - 9X_B - 4X_C &= 0 \dots\dots\dots i \\ -5X_A + 12X_B - 10X_C &= 0 \dots\dots\dots ii \\ -9X_A - 3X_B + 14X_C &= 0 \dots\dots\dots iii \end{aligned}$$

Solving for X_A by elimination using equation (i) and (ii).

$$\begin{aligned} 5(14X_A - 9X_B - 4X_C &= 0) \\ 14(-5X_A + 12X_B - 10X_C &= 0) \\ \\ 70X_A - 45X_B - 20X_C &= 0 \\ \Rightarrow -70X_A + 168X_B - 140X_C &= 0 \\ \hline 123X_B - 160X_C &= 0 \dots\dots\dots iv \end{aligned}$$

Solving for X_A by elimination using equation (i) and (iii).

$$\begin{aligned} 9(14X_A - 9X_B - 4X_C &= 0) \\ 14(-9X_A - 3X_B + 14X_C &= 0) \\ \\ 126X_A - 81X_B - 36X_C &= 0 \\ \Rightarrow -126X_A - 42X_B + 196X_C &= 0 \\ \hline -123X_B + 160X_C &= 0 \dots\dots\dots v \end{aligned}$$

From equation (iv) and (v)

$$X_B = \frac{160}{123} X_C$$

But $X_C = 1230$

$$\text{Hence, } X_B = \frac{160}{123} (1230) = 1600$$

From equation (i)

$$\begin{aligned} X_A &= \frac{9X_B + 4X_C}{14} \\ &= \frac{9(1600) + 4(1230)}{14} = 1380 \end{aligned}$$

Therefore, the production of industry A is 1380, and that of industry B is 1600.

Bibliography

Lucey, T. (2002). *Quantitative Techniques* (6th ed.). Cengage Learning.

Taha, H. A. (2017). *Operation Research An introduction* (10th ed.). Prentice-Hall, Inc.