OPERATIONS RESEARCH

LECTURE FOUR

Markov chains

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INTRODUCTION

This lecture will focus on probability transition matrix and solving problems involving Markov chain.

Intended learning outcomes

At the end of this lecture, you will be able to

i. Define Markov chain
ii. Explain properties of a Markov Chain
iii. Formulate transition probability matrix
iv. Solve steady state conditions for a business problem

References

These lecture notes should be supplemented with relevant topics from the books listed in the Bibliography at the end of the lecture

Transition matrix and State matrix

These are matrices in which the individual elements are in the form of probabilities.

Transition matrix:

Many types of application involve a set of states $\{S_1, S_2, \ldots, S_n\}$ of given population for instance residence of a city may live downtown or I suburbs, soft drink customers may buy pepsi-cola, cocacola or another brand and so on.

Members of a given population will change state. The probability that a member of a population will change state. The probability that a member of population will change from the j^{th} state to the i^{th} state is given by P_{ij} wher P_{ij} is a number between 0 and 1. i.e., $0 \leq P_{ij} \leq 1$.

In particular P_{ii} is the probability that a member of population will remain in the i^{th} state. If the probability $P_{ij} = 0$, a member is certain not to change from the j^{th} state to the i^{th}. $P_{ij} = 1$ means that the member is certain to move from the j^{th} state to the i^{th} state.

The collection of all probabilities P_{ij} is represented by a square matrix P as:

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1n} \\ P_{21} & P_{22} & P_{23} & \cdots & P_{2n} \\ P_{31} & P_{32} & P_{33} & \cdots & P_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & P_{n3} & \cdots & P_{nn} \end{pmatrix} \begin{matrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_n \end{matrix} \quad To$$
$$\begin{matrix} S_1 & S_2 & S_3 & \cdots & S_n \end{matrix}$$
$$From$$

The matrix P is called the transition matrix. The sum of elements in each column of P is one. A matrix that has this property is called a Stochastic matrix.

A stochastic matrix is a matrix P that has each element as a number between 0 and 1 i.e., $0 \leq P \leq 1$, and the sum of elements in each column is one.

Example

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \qquad B = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}; \qquad C = \begin{pmatrix} 0.1 & 0.8 & 1 \\ 0.2 & 0.1 & 0 \\ 0.7 & 0.2 & 0.1 \end{pmatrix}$$

A and B are stochastic matrices because each of their elements are non-negative and they sum to 1 in each column.

C is not stochastic matrix because the elements in the 2^{nd} and 3^{rd} column do not sum to one.

State matrix:

A state matrix of population is a column matrix whose entries represent portions of the total population that are in each of the various states.

For example, consider a population of 10,000 with the following states

	State	Number in state
S_1	Under 21 years old	$2,500$
S_2	$21-65$ years old	$4,500$
S_3	Over 65 years old	$3,000$

The portions of the total population in each state will be given as:

$$S_1 = \frac{2,\ 500}{10,000} = 0.25; \qquad S_2 = \frac{4,500}{10,000} = 0.45; \qquad S_3 = \frac{3,000}{10,000} = 0.3$$

The state matrix therefore becomes;

$$X = \begin{pmatrix} 0.25 \\ 0.45 \\ 0.3 \end{pmatrix}$$

In general, a state matrix of population with n states having portions of $X_1, X_2, X_3, \dots, X_n$ is

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{pmatrix}$$

Where $X_i =$ fraction/portion of; $X_1 + X_2 + X_3 + \cdots + X_n = 1$

Multiplying a transition matrix by a state matrix, we obtain a new state matrix. This new state matrix gives the portions of the total population that will be in each of the various states after one transition

$$PX = \begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1n} \\ P_{21} & P_{22} & P_{23} & \cdots & P_{2n} \\ P_{31} & P_{32} & P_{33} & \cdots & P_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & P_{n3} & \cdots & P_{nn} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{pmatrix}$$

Transition matrix state matrix New state matrix

Example

Two grocery stores in Sabaab compete for the same customers. Although each store has a few loyal customers, some shop in all stores. After conducting a survey, the management of one of the stores has determined that 40% of the people who shop at store A will shop at the same store, while 60% will go to store B. For store B, 80% of the people will reurn to the same store while 20% will go to store A. If 3,000 people shoped at A and 5,000 people shoped at B this week, how many will shop at each store next week?

Solution

$$P = \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \qquad X_0 = \begin{pmatrix} \frac{3}{8} \\ \frac{5}{8} \end{pmatrix}$$

$$\text{New state} = PX_0 = \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} 0.375 \\ 0.625 \end{pmatrix} = \begin{pmatrix} 0.275 \\ 0.725 \end{pmatrix}$$

The number of people who will shop at the different stores next week will be:

$$A = 0.275 \times 8{,}000 = 2{,}200$$

$$B = 0.725 \times 8{,}000 = 5{,}800$$

Definition

Markov chain, named after the Russian mathematician Andrew Markov, is an experiment consisting of the sequence of trials for which an outcome of each trial depends on the outcome of the previous trial

Markov processes are defined as a set of trials which follow a certain sequence depending on the transition probabilities. These probabilities indicate how a particular activity or product moves from one state to another.

i.e.

$$X_0 = \begin{pmatrix} 0.375 \\ 0.625 \end{pmatrix}; \qquad X_1 = PX_0 = \begin{pmatrix} 0.275 \\ 0.725 \end{pmatrix}; \qquad X_2 = PX_1$$

Illustration

From the previous example, determine the customer numbers in each state after 3 weeks.

$$X_1 = PX_0 = \begin{pmatrix} 0.275 \\ 0.725 \end{pmatrix}; \qquad X_2 = PX_1 = \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} 0.275 \\ 0.725 \end{pmatrix} = \begin{pmatrix} 0.255 \\ 0.745 \end{pmatrix};$$

$$X_3 = \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} 0.255 \\ 0.745 \end{pmatrix} = \begin{pmatrix} 0.251 \\ 0.749 \end{pmatrix}$$

After 3 weeks;

$$A = 0.251 \times 8{,}000 = 2{,}008 \text{ customers}$$

$$B = 0.749 \times 8{,}000 = 5{,}992 \text{ customers}$$

N^{th} state Markov chain

Let P be the transition matrix and X be the state matrix. The n^{th} state Markov chain is given by

$$X_n = P^n X_0$$

In the previous example, find the 2^{nd} state matrix (X_2)

$$X_2 = P^2 X_0 = \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} 0.375 \\ 0.625 \end{pmatrix}$$

$$= \begin{pmatrix} 0.28 & 0.24 \\ 0.72 & 0.76 \end{pmatrix} \begin{pmatrix} 0.375 \\ 0.625 \end{pmatrix} = \begin{pmatrix} 0.255 \\ 0.745 \end{pmatrix}$$

Example

The market research department for a manufacturing plant has determined that a portion of the population will purchase their product during any given month. It further found out that of the people who purchased from them, 20% of them will not purchase next month while 30% who do not purchase their product at any given month will purchase next month. In a population of 10,000 people, 1,000 people purchased the product this month. How many will purchase the product in 3 months?

Solution

$$P = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \qquad X_0 = \begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}\begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix} = \begin{pmatrix} 0.35 \\ 0.65 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}\begin{pmatrix} 0.35 \\ 0.65 \end{pmatrix} = \begin{pmatrix} 0.475 \\ 0.525 \end{pmatrix}$$

$$X_3 = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}\begin{pmatrix} 0.475 \\ 0.525 \end{pmatrix} = \begin{pmatrix} 0.5375 \\ 0.4625 \end{pmatrix}$$

Therefore, those who will purchase $= 0.5375 \times 10{,}000 = 5375$

Regular Markov chain

A stochastic matrix P is called a regular matrix if sme powers of P have only positive entries. This implies that all the elements in P^n are all positive for $n = 1, 2, 3, \ldots$

Example

i. $P = \begin{pmatrix} 0.5 & 0.25 \\ 0.5 & 0.75 \end{pmatrix}$

P is regular since it is stochastic and P^1 has only positive entries.

ii. $P = \begin{pmatrix} 0.5 & 1 \\ 0.5 & 0 \end{pmatrix}$; $\qquad P^2 = \begin{pmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{pmatrix}$

P is regular because it is stochastic and P^2 has only positive entries.

iii. $P = \begin{pmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{pmatrix}$

P is not regular since it is not stochastic (sum of elements in its columns is not equal to 1).

A regular Markov chain is concerned with the long-term trends or steady states after many trials.

If P is a regular matrix then the sequence $P^1, P^2, P^3, P^4, \ldots, P^n$ approaches a stable matrix $\bar{P}$.

The sequence $P^1X,\ P^2X,\ P^3X,\ P^4X,\ \ldots$ approaches $\bar{X}$, where $\bar{X}$ is called the stable distribution matrix.

The elements in each column of $\bar{P}$ are equal to the corresponding elements in the column of matrix $\bar{X}$. To find $\bar{X}$, solve the equation

$$P\bar{X} = \bar{X}$$

Example

Find the stable distribution matrix and stable matrix if $P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix}$

Solution

Let $\bar{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \Longrightarrow \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

$$\therefore \begin{matrix} \frac{I}{2}X_1 + \frac{1}{4}X_2 = X_1 \\ \frac{1}{2}X_1 + \frac{3}{4}X_2 = X_2 \end{matrix} \qquad \Longrightarrow \qquad \begin{matrix} -\frac{I}{2}X_1 + \frac{1}{4}X_2 = 0 \\ \frac{1}{2}X_1 - \frac{1}{4}X_2 = 0 \end{matrix}$$

$$\frac{1}{4}X_2 = \frac{1}{2}X_1$$

$$X_2 = 2X_1$$

$$\text{But,} \qquad \mathrm{X}_1 + \mathrm{X}_2 = 1$$

$$\mathrm{X}_1 + 2\mathrm{X}_1 = 1 \qquad \Longrightarrow X_1 = \frac{1}{3}; \qquad X_2 = \frac{2}{3}$$

$$\bar{X} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \qquad\qquad \bar{P} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

Example

Let $P = \begin{pmatrix} \frac{3}{4} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{8} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}$, find the stable distribution matrix.

Solution

$$P\bar{X} = \bar{X}$$

$$\begin{pmatrix} \frac{3}{4} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{8} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}\bar{X} = \bar{X}$$

$$\begin{aligned} \frac{3}{4}X_1 + \frac{1}{3}X_2 + \frac{1}{3}X_3 &= X_1 \\ \frac{1}{8}X_1 + \frac{1}{3}X_2 + \frac{1}{2}X_3 &= X_2 \\ \frac{1}{8}X_1 + \frac{1}{3}X_2 + \frac{1}{6}X_3 &= X_3 \end{aligned}$$

$$\begin{aligned} -\frac{1}{4}X_1 + \frac{1}{3}X_2 + \frac{1}{3}X_3 &= 0 \\ \frac{1}{8}X_1 - \frac{2}{3}X_2 + \frac{1}{2}X_3 &= 0 \\ \frac{1}{8}X_1 + \frac{1}{3}X_2 - \frac{5}{6}X_3 &= 0 \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} -3X_1 + 4X_2 + 4X_3 &= 0 \\ 3X_1 - 16X_2 + 12X_3 &= 0 \\ 3X_1 + 8X_2 - 20X_3 &= 0 \end{aligned}$$

Eliminating X_1 using equation (i)

$$\begin{aligned} -12X_2 + 16X_3 &= 0 \\ 12X_2 - 16X_3 &= 0 \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} 12X_2 &= 16X_3 \\ X_2 &= \frac{4}{3}X_3 \end{aligned}$$

Substituting for X_2 in equation (i)

$$-3x_1 + \frac{16}{3}X_3 + 4X_3 = 0$$

$$3X_1 = \frac{28}{3}X_3 \quad \Rightarrow \quad X_1 = \frac{28}{9}X_3$$

But $X_1 + X_2 + X_3 = 1$

$$\frac{28}{9}X_1 + \frac{4}{3}X_2 + X_3 = 1 \quad \Rightarrow \quad \frac{49}{9}X_3 = 1 \quad \therefore \quad X_3 = \frac{9}{49}$$

$$X_1 = \frac{28}{49}; \qquad X_2 = \frac{12}{49}; \qquad X_3 = \frac{9}{49}$$

Thus $\bar{X} = \begin{pmatrix} \frac{28}{49} \\ \frac{12}{49} \\ \frac{9}{49} \end{pmatrix}$ $\qquad \bar{P} = \begin{pmatrix} \frac{28}{49} & \frac{28}{49} & \frac{28}{49} \\ \frac{12}{49} & \frac{12}{49} & \frac{12}{49} \\ \frac{9}{49} & \frac{9}{49} & \frac{9}{49} \end{pmatrix}$

Areas of application of Markov chains

The Markov processes or chains are frequently applied as follows: -

Brand Switching - By using the transitional probabilities, we can be able to express the way consumers switch their tastes from one product to another.

Insurance industry - Markov analysis may be used to study the claims made by the insured persons and decide the level of premiums to be paid in future.

Movement of urban population - By formulating a transition matrix for the current population in the urban areas, one can be able to determine what the population will be in say 5 years.

Movement of customers from one bank to another - Customers tend to look for efficient banks. Therefore, at a certain time when a given bank improve on their customer satisfaction it will tend to attract several customers who will move from certain banks to more efficient ones.

Bibliography

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