OPERATIONS RESEARCH

LECTURE FIVE

Linear programming (1)

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INTRODUCTION

This lecture will focus on definition of terms and formulation of linear programming models

Intended learning outcomes

At the end of this lecture, you will be able to

i. Explain the properties of Linear Programming Model
ii. Formulate linear programming models

References

These lecture notes should be supplemented with relevant topics from the books listed in the Bibliography at the end of the lecture

Definition of Operations Research

Operations Research is a scientific methodology which is applied to the study of observations of large and/or complex organizations or activities to assess the overall implications of various alternative courses of action to provide an improved basis for management decisions.

An operations research worker is required to:

- Minimize the input for a specific output and/or
- Maximize the output value for a scientific input and/or
- Maximize some function of these values for instance, the profit function, return on investment function etc.

The structure of mathematical models

Model: -

A model in the sense used in operations research is defined as an idealized representation of a real-life system. A mathematical model includes mainly three basic sets of elements namely:

1. **Decision variables and parameters –** a variable is a quantity which takes different values at different times. Decision variables are therefore the unknowns that are to be determined.
2. **Constraints or restrictions –** these are the conditions that limit the decision variables to their feasible (or permissible) values.

   Illustration:

   Let x_1 and x_2 be the number of units to be produced of two products (decision variables) and let a_1 and a_2 be their respective per unit requirements of raw material (parameters). If the total available of the raw material is A, then the corresponding constraint function is given by

   $$a_1x_1 + a_2x_2 \leq A$$

   Another common constraint is the non-negativity constraint which requires that all decision variables be either zero or positive. In the above illustration, they would be $x_1 \geq 0; \quad x_2 \geq 0$
3. **Objective function –** this defines the measure of effectiveness of the system as a mathematical function of its decision variables. It acts as an indicator for the achievement of the optimum solution.

The general Linear programming problem

The general linear programming problem calls for optimizing (maximizing or minimizing) the linear function of variables called the objective function, subject to a set of linear equalities and/or inequalities called constraints or restrictions.

Example

Consider the situation of deciding on the number of units to be manufactured of two different products. Let the profits per unit of product 1 and product 2 be 2 an 5 respectively.

Each unit of product 1 requires 3 machine hours and 9 units of raw material while each unit of product 2 requires 4 machine hours and 7 units of raw material. The maximum available machine hours and raw material units are 200 and 300 respectively. A minimum of 20 units is required for product 1.

This problem can be formulated mathematically as follows:

Let x_1 and x_2 be the decision variables corresponding to product 1 and product 2 respectively. The information can be thus summarized as:

	Product 1	Product 2
Number of units produced (variables)	x_1	x_2
Profit	$2x_1$	$5x_2$
Number of machine hours	$3x_1$	$4x_2$
Units of raw material	$9x_1$	$7x_2$
Minimum requirement	20	0

The objective of this problem is to maximize the total profit $2x_1 + 5x_2$ for the two products.

This maximization is subject to the machine hours constraints, the raw material constraint, and the minimum requirement constraint.

The machine hours constraint specifies that the total number of machine hours used by the two products should not exceed maximum number of hours available i.e.

$$3x_1 + 4x_2 \leq 200$$

Similarly for the raw material constraint,

$$9x_1 + 7x_2 \leq 300$$

And for the minimum requirements constraints

$$x_1 \geq 20; \qquad x_2 \geq 0$$

The problem is usually put in the format:

Maximize $x_0 = 2x_1 + 5x_2$

$$\text{Subject to:} \quad \begin{array}{l} 3x_1 + 4x_2 \leq 200 \\ 9x_1 + 7x_2 \leq 300 \\ x_1 \geq 20; \quad x_2 \geq 0 \end{array}$$

where x_0 is the objective function.

The above example illustrates three main properties of the general linear programming problem:

1. The objective function and the constraints are linear expressions of the decision variables.
2. Each constraint may be one of the three types:
    - Less than or equal to ($\leq$)
    - Greater than or equal to ($\geq$)
    - Equal to ($=$)

The last type is not illustrated in the above example. However, if a market restriction requires that the ratio of product 1 to product 2 be equal to 6, then an additional constraint $\frac{x_1}{x_2} = 6$ or $x_1 - 6x_2 = 0$ should be added to the above problem.

3. All the variables of the linear programming problem are non-negative.

Formulation of linear optimization models

The objective is to familiarize ourselves with some of the areas where the above technique may be applicable. We shall use examples to stress mainly the formulation aspect of the problem rather than its solution aspect. Later we shall present the general method for solving any liner programming problem.

Example 1

A textile factory in a certain town produces three different grades of cloth all of which are wool/silk mixture. The number of units of wool and silk needed to make one unit of each type of cloth together with profit to be made from each unit is given below:

	Wool	Silk	Profit
Type 1	3	2	4
Type 2	1	1	1
Type 3	4	3	5

The maximum amount of raw wool or silk that may be used each week by the factory is determined by the current limits being:

Wool 8,000 units
Silk 3,000 units

The factory manager is required to maximize his profits and wishes to know how to achieve this.

Solution

Let $x_i, \quad i = 1,2,3$ be the number of units of type 1,2,3 woven.

Objective is to maximize profit, thus;

Maximize $x_0 = 4x_1 + \ x_2 + 5x_3$

$$\text{Subject to} \quad \begin{aligned} 3x_1 + x_2 + 4x_3 &\leq 8{,}000 \\ 2x_1 + x_2 + 3x_3 &\leq 3{,}000 \\ x_1 \geq 0; \quad x_2 \geq 0; \quad x_3 &\geq 0 \end{aligned}$$

Example 2

A pig breeder requires feed weanes as quickly as possible. They require a diet consisting of a minimum of 3 nutrients $N_1, \ N_2, \ N_3$. These nutrients form part of 4 commercial food stuffs $f_1, \ f_2, \ f_3, \ f_4$.

	Nutrients /unit food				Minimum required
	f_1	f_2	f_3	f_4	
N_1	5	8	4	1	50
N_2	3	8	7	5	42
N_3	4	0	5	4	8
Cost/unit	10	9	12	9	

Form a linear optimization model (LOM).

Solution

Let $x_1,\ x_2,\ x_3,\ x_4$ be the amount of $f_1,\ f_2,\ f_3,\ f_4$ to be purchased.

Objective

Minimize $x_0 = 10x_1 + 9x_2 + 12x_3 + 9x_4$

Subject to
$$\begin{array}{c} 5x_1 + 8x_2 + 4x_3 + x_4 \geq 50 \\ 3x_1 + 8x_2 + 7x_3 + 5x_4 \geq 42 \\ 4x_1 + 0x_2 + 5x_3 + 4x_4 \geq 8 \\ x_1 \geq 0,\ \ x_2 \geq 0,\ \ x_3 \geq 0,\ \ x_4 \geq 0 \end{array}$$

In general, the linear programming problem can now be defined as follows:

Maximize (or minimize) $x_0 = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Subject to
$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ \leq,\ =\ \ or\ \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \ \leq,\ =\ \ or\ \geq b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \ \leq,\ =\ \ or\ \geq b_m \\ x_1 \geq 0,\ \ x_2 \geq 0,\ \ \ldots,\ \ x_n \geq 0 \end{array}$$

Where $c_j,\ b_i$ and a_{ij} $(i = 1, 2, \ldots, m;\ \ j = 1, 2, \ldots, n)$ are constants which are determined depending on the technology of the problem and $x_j's$ are the decision variables.

Noted that for each constraint, only one of the signs $(\leq, \ = \ \ or \ \geq)$ holds.

The above formulation may be put in the following compact form by using the summation sign:

Maximize (or minimize) $x_0 = \sum_{j=1}^{n} c_j x_j$

Subject to $\sum_{j=1}^{n} a_{ij} x_j \quad \leq, \ = \ \ or \ \geq b_i \qquad i = 1, 2, \dots, \ m$

$$x_j \geq 0 \qquad j = 1, \ 2, \dots, \ n$$

In matrix form we have;

Maximize (or minimize) $x_0 = (c_1 \quad c_2 \quad \cdots \quad c_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

Subject to $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} \leq \\ = \\ \geq \end{matrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

$$x_1 \geq 0, \ \ x_2 \geq 0, \ \ \dots, \ \ x_n \geq 0$$

Bibliography

Lucey, T. (2002). *Quantitative Techniques* (6th ed.). Cengage Learning.

Taha, H. A. (2017). *Operation Research An introduction* (10th ed.). Prentice-Hall, Inc.