OPERATIONS RESEARCH

LECTURE SEVEN

Simplex method (1)

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INTRODUCTION

This lecture will focus on definition of terms used in Simplex method and Introduction to the simplex tableau.

Intended learning outcomes

At the end of this lecture, you will be able to explain basic terms in Simplex analysis

References

These lecture notes should be supplemented with relevant topics from the books listed in the Bibliography at the end of the lecture

SOLVING MATHEMATICAL MODELS FOR LINEAR PROGRAMMING

There are two main methods of solving linear programming problems. In this lecture, we will introduce the second method which is the Simplex method.

This method is used to solve linear programming problems with two or more decision variables.

Definition of terms

- Iteration – sequence of steps performed in moving from one basic feasible solution to another.
- Simplex tableau – Simplex table.
- Pivot column – Column of the simplex tableau with the most negative indicator.
- Pivot row – Row of the simplex tableau with the least non-negative quotient.
- Pivot element – Element at the intersection of the pivot row and pivot column.

Procedure

1. Determine the objective.
2. Write all the necessary constraints.
3. Transform the optimization model into standard form. The characteristics of the standard form are:
   a. All constraints are equations except for the non-negativity constraints which remain inequalities (≥ 0).

      Constraints of the inequality type can be changed to equations by adding or subtracting from the left-hand side of each such constraint a non-negative variable. These new variables are called SLACK variables.

      The slack variables are added if the constraint is $\leq$ or subtracted if the constraint is $\geq$.

      The slack variable constraint can be illustrated as below:

      Consider the constraint

      $$a_1x_1 + a_2x_2 \geq b\ , \qquad b \geq 0$$

      This is transformed to standard form to be

      $$a_1x_1 + a_2x_2 - s_1 = b; \qquad s_1 \geq 0$$

      Also, the constraint

      $$p_1x_1 + p_2x_2 \leq q\ , \qquad q \geq 0$$

      can be transformed to standard form as

      $$p_1x_1 + p_2x_2 + s_2 = q; \qquad s_2 \geq 0$$

   b. The right-hand side element of each element of each constraint equation is non-negative.

      If the right-hand side of a constraint is negative, it can be made positive by multiplying both sides of the result equation by -1.

   c. All variables are non-negative
   d. The objective function is of maximization or minimization type.

Example

Express the following linear optimization models in standard form.

i. Maximize $y_o = 6y_1 + 7y_2$

$$\text{Subject to} \quad \begin{array}{c} 2y_1 + 3y_2 \leq 24 \\ 2y_1 + y_2 \leq 16 \\ y_1,\ y_2 \geq 0 \end{array}$$

Solution

The standard for will be expressed as below

Maximize $y_o - 6y_1 - 7y_2 = 0$

$$\text{Subject to} \quad \begin{array}{c} 2y_1 + 3y_2 + s_1 = 24 \\ 2y_1 + y_2 + s_2 = 16 \\ y_1,\ y_2,\ s_1,\ s_2 \geq 0 \end{array}$$

ii. Maximize $x_0 = 3x_1 + 2x_2 + 3x_3$

$$\text{Subject to} \quad \begin{array}{c} 2x_1 - 3x_2 \leq 3 \\ x_1 + 2x_2 + 3x_3 \geq 5 \\ 3x_1 + 2x_3 \leq 2 \\ x_1,\ x_2,\ x_3 \geq 0 \end{array}$$

Solution

In standard form, this model becomes

Maximize $x_0 - 3x_1 - 2x_2 - 3x_3 = 0$

$$\text{Subject to} \quad \begin{array}{c} 2x_1 - 3x_2 + s_1 = 3 \\ x_1 + 2x_2 + 3x_3 - s_2 = 5 \\ 3x_1 + 2x_3 + s_3 = 2 \\ x_1,\ x_2,\ x_3,\ s_1,\ s_2,\ s_3 \geq 0 \end{array}$$

4. Set up the initial simplex tableau. This is a useful way of presenting the information of the standard form in preparation for the solution.

Illustration

Consider the linear optimization model with all less than or equal to, $(\leq)$, type of constraints.

Maximize $x_0 = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Subject to
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \\ x_1 \ge 0,\ \ x_2 \ge 0,\ \ \ldots,\ \ x_n &\ge 0 \end{aligned}$$

When expressed in standard form it becomes

Maximize $x_0 - c_1x_1 - c_2x_2 - \cdots - c_nx_n = 0$

Subject to
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + s_1 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + s_2 &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + s_m &= b_m \end{aligned}$$
$$x_1 \ge 0,\ \ x_2 \ge 0,\ \ \ldots,\ \ x_n \ge 0,\ \ s_1 \ge 0,\ \ s_2 \ge 0,\ \ \ldots,\ \ s_m \ge 0$$

The initial simplex tableau then becomes

Basic solution	x_0	x_1	x_2	...	x_n	s_1	s_2	...	s_m	Solution
x_0	1	$-c_1$	$-c_2$	...	$-c_n$	0	0	...	0	0
s_1	0	a_{11}	a_{12}	...	a_{1n}	1	0	...	0	b_1
s_2	0	a_{21}	a_{21}	...	a_{2n}	0	1	...	0	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s_m	0	a_{m1}	a_{m2}	...	a_{mn}	0	0	...	1	b_m

Optimality and feasibility conditions of the Simplex method

5. The entering variable is a non-basic variable with the most negative indicator. If there exists a tie in the variables, the entering variable is chosen arbitrarily. The column of the simplex tableau corresponding to the entering variable is the PIVOT column
6. Form the necessary quotients to find the PIVOT row, hence the leaving variable as the variable corresponding to the pivot row.

   The quotients are determined as the ratios of the current basic solutions to the corresponding coefficients of the entering variable.

Any negative quotient or quotient with a zero denominator is to be disregarded. If all quotients are to be disregarded, no optimal solution exists.

The smallest non-negative quotient gives the location of the pivot row. If a tie exists in the smallest non-negative quotient, let either determine the pivot row.

7. After determining the entering and leaving variable, transform the tableau so that the PIVOT element becomes 1 and all other elements in the pivot column become zeros. This transformation is achieved by applying Gauss-Jordan method of solving systems of equations. There are two sets of computation that will be involved:

   To transform the pivot element to 1,

$$\text{New pivot equation} = \frac{\text{old pivot equation}}{\text{pivot element}}$$

   To transform all the other elements in the pivot column to 0,

$$\text{New equation} = \text{old equation} \pm \begin{pmatrix}\text{corresponding coefficient} \\ \text{of entering variable}\end{pmatrix} \times \begin{pmatrix}\text{new pivot} \\ \text{equation}\end{pmatrix}$$

8. If all the indicators on the indicator row are all positive or zero(0), this is the final tableau. If not, go back to step 5 above and repeat the process until a tableau with no negative indicator is obtained.
9. Read the solution from the right-hand side of this final tableau. The optimal value of the objective function is the number at the top right corner of the final tableau

<u>Illustration</u>

Set up the initial tableau for the following linear optimization models

a. Maximize $x_0 = 9x_1 + 10x_2$

$$\text{Subject to } \begin{aligned} x_1 + 2x_2 &\leq 8 \\ 5x_1 + 2x_2 &\leq 16 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

b. Maximize $x_0 = 3x_1 + 5x_2 - 2x_3$

$$\text{Subject to } \begin{aligned} x_1 + 2x_2 + 2x_3 &\leq 10 \\ 2x_1 + 4x_2 + 3x_3 &\geq 15 \\ x_1,\ x_2,\ x_3 &\geq 0 \end{aligned}$$

Solution

a. Maximize $x_0 = 9x_1 + 10x_2$

$$\text{Subject to } \begin{aligned} x_1 + 2x_2 &\leq 8 \\ 5x_1 + 2x_2 &\leq 16 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

In standard form this model becomes:

Maximize $x_0 - 9x_1 - 10x_2 = 0$

$$\text{Subject to } \begin{aligned} x_1 + 2x_2 + s_1 &= 8 \\ 5x_1 + 2x_2 + s_2 &= 16 \\ x_1,\ x_2,\ s_1,\ s_2 &\geq 0 \end{aligned}$$

Then the initial tableau is:

Basic solution	x_0	x_1	x_2	s_1	s_2	Solution
x_0	1	-9	-10	0	0	0
s_1	0	1	2	1	0	8
s_2	0	5	2	0	1	16

b. Maximize $x_0 = 3x_1 + 5x_2 - 2x_3$

$$\text{Subject to } \begin{aligned} x_1 + 2x_2 + 2x_3 &\leq 10 \\ 2x_1 + 4x_2 + 3x_3 &\geq 15 \\ x_1,\ x_2,\ x_3 &\geq 0 \end{aligned}$$

This model is expressed in standard form to become

Maximize $x_0 - 3x_1 - 5x_2 + 2x_3 = 0$

$$\text{Subject to } \begin{aligned} x_1 + 2x_2 + 2x_3 + s_1 &= 10 \\ 2x_1 + 4x_2 + 3x_3 - s_2 &= 15 \\ x_1,\ x_2,\ x_3,\ s_1,\ s_2 &\geq 0 \end{aligned}$$

The corresponding initial tableau is

Basic solution	x_0	x_1	x_2	x_3	s_1	s_2	Solution
x_0	1	-3	-5	2	0	0	0
s_1	0	1	2	2	1	0	10
s_2	0	2	4	3	0	-1	15

Bibliography

Lucey, T. (2002). *Quantitative Techniques* (6th ed.). Cengage Learning.

Taha, H. A. (2017). *Operation Research An introduction* (10th ed.). Prentice-Hall, Inc.