OPERATIONS RESEARCH

LECTURE EIGHT

Simplex method (2)

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INTRODUCTION

This lecture will focus on solving linear programming problems using Simplex method.

Intended learning outcomes

At the end of this lecture, you will be able to solve linear programming problems using the simplex method.

References

These lecture notes should be supplemented with relevant topics from the books listed in the Bibliography at the end of the lecture

Example 1

Use the simplex method to find the optimal solution to the linear programming model below

Maximize $x_0 = 9x_1 + 10x_2$

$$\text{Subject to } \begin{aligned} x_1 + 2x_2 &\leq 8 \\ 5x_1 + 2x_2 &\leq 16 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

Solution

In standard form this model becomes:

Maximize $x_0 - 9x_1 - 10x_2 = 0$

$$\text{Subject to } \begin{aligned} x_1 + 2x_2 + s_1 &= 8 \\ 5x_1 + 2x_2 + s_2 &= 16 \\ x_1,\ x_2,\ s_1,\ s_2 &\geq 0 \end{aligned}$$

Then the initial tableau is:

Basic solution	x_0	x_1	x_2	s_1	s_2	Solution
x_0	1	-9	-10	0	0	0
s_1	0	1	2	1	0	8
s_2	0	5	2	0	1	16

A starting basic solution can be obtained directly using the slack variables. This gives $s_1 = 8$ and $s_2 = 16$. Since all the non-basic variables x_1 and $x_2 = 0$. Also the objective function $x_0 = 0$.

Note

- The column under x_0 contains a unit element support in the first row and zero elements otherwise
- The columns under s_1 and s_2 similarly contain unit elements in the second and third rows respectively.

Due to this arrangement, the right-hand side of the tableau yields the values of the starting solution. We shall be using this special structure of a starting tableau. Coefficient of the starting basic solution in the indicator row must be zero.

After all the above conditions, the entering variable is selected as a non-basic variable having the most negative indicator in the indicator row.

In the above example x_2 is the entering variable. One of the current basic variables s_1 or s_2 must leave the solution and become non-basic at zero level. A determination of the specific leaving variable is achieved by using feasibility condition.

We take the ratios of the values of the current basic variables excluding x_0, to the corresponding constraints coefficients of the entering variable. The leaving variable then corresponds to the least non-negative quotient. i.e.,

Current basic solution	Ratio to coefficients of x_2
$s_1 = 8$	$\frac{8}{2} = 4$
$s_2 = 16$	$\frac{16}{2} = 8$

Hence s_1 becomes the leaving variable as x_2 enters the solution.

After identifying the entering variable x_2 and the leaving variable s_1, the next step is to modify the previous tableau to directly give the solution of the new basic variable. This is done by eliminating the entering variable (x_2 in this example) from the indicator row (objective equation) and from all the constraint equations except where the leaving variable (s_1 in this example) appears. This implies the objective equation will have only x_1 and s_1.

From the previous tableau

Basic solution	x_0	x_1	x_2	s_1	s_2	Solution
x_0	1	−9	−10	0	0	0
s_1	0	1	②	1	0	8
s_2	0	5	2	0	1	16

The coefficient ② of the entering variable corresponding to the minimum ratio is called the PIVOT element. For the right-hand side of the first constraint equation to yield the value of x_2, the whole equation must be divided by the pivot element (= 2 in this example). This row which is the pivot equation appears in the new tableau as follows

Basic solution	x_0	x_1	x_2	s_1	s_2	Solution
x_2	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	4

The elimination of x_2 from the x_0 equation and the second constraint equation is equivalent to creating zero coefficients for x_2 in these two equations.

This is done using the following row operations.

1. Multiply the new pivot equation by 10 and add the result to the x_0 equation.
2. Multiply the new pivot equation by -2 and add the results to the second constraint. This gives the new the new tableau which represents the start of the new iteration.

Basic solution	x_0	x_1	x_2	s_1	s_2	Solution
x_0	1	-4	0	5	0	40
x_2	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	4
s_2	0	4	0	-1	1	8

This tableau reveals that the current tableau is non-optimal since the coefficient of x_1 in the indicator row is negative hence, applying the same conditions, x_1 should enter the solution.

The leaving variable is determined as follows

Current basic solution	Ratio to coefficients of x_2
$x_2 = 4$	$\frac{4}{1/2} = 8$
$s_2 = 8$	$\frac{8}{4} = 2$

s_2 is the leaving variable.

The new tableau is therefore,

Basic solution	x_0	x_1	x_2	s_1	s_2	Solution
x_0	1	0	0	4	1	48
x_2	0	0	1	$\frac{5}{8}$	$-\frac{1}{8}$	3
x_1	0	1	0	$-\frac{1}{4}$	$\frac{1}{4}$	2

Since all the coefficients in the indicator row are all positive, this is the optimal solution. Thus;

$$x_1 = 2, \quad x_2 = 3, \quad s_1 = s_2 = 0, \quad x_0 = 48.$$

The constraint whose slack variable is zero is the scarce or limited resource which is referred to as BINDING constraint.

The dual price of the binding constraint is the value of the slack variables in the indicator row of the final tableau.

Example 2

Consider the following linear programming problem.

Maximize $x_0 = 3x_1 + 2x_2 + 5x_3$

Subject to
$$x_1 + 2x_2 + x_3 \leq 430$$
$$3x_1 + 2x_3 \leq 460$$
$$x_1 + 4x_2 \leq 420$$
$$x_1 \geq 0, \;\; x_2 \geq 0, \;\; x_3 \geq 0$$

Find the optimal solution.

Solution

Since there are three decision variables in this problem, the appropriate method for solving is the simplex method.

In standard form the model will be:

Maximize $x_0 - 3x_1 - 2x_2 - 5x_3 = 0$

Subject to
$$x_1 + 2x_2 + x_3 + s_1 = 430$$
$$3x_1 + 2x_3 + s_2 = 460$$
$$x_1 + 4x_2 + s_3 = 420$$
$$x_1 \geq 0, \;\; x_2 \geq 0, \;\; x_3 \geq 0, \;\; s_1 \geq 0, \;\; s_2 \geq 0, \;\; s_3 \geq 0$$

The initial tableau is

Basic solution	x_0	x_1	x_2	x_3	s_1	s_2	s_3	Solution
x_0	1	-3	-2	-5	0	0	0	0
s_1	0	1	2	1	1	0	0	430
s_2	0	3	0	2	0	1	0	460
s_3	0	1	4	0	0	0	1	420

First iteration

x_3 is the entering variable. Taking ratios

Current basic solution	Ratio to coefficients of x_2
$s_1 = 430$	$\frac{430}{1} = 430$
$s_2 = 460$	$\frac{460}{2} = 230$
$s_3 = 420$	$\frac{420}{0} \Rightarrow$ ignore

s_2 is the leaving variable. The new tableau is

Basic solution	x_0	x_1	x_2	x_3	s_1	s_2	s_3	Solution
x_0	1	$9/2$	-2	0	0	$5/2$	0	1150
s_1	0	$-1/2$	2	0	1	$-1/2$	0	200
x_3	0	$3/2$	0	1	0	$1/2$	0	230
s_3	0	1	4	0	0	0	1	420

Second iteration

x_2 is the entering variable. Taking ratios,

Current basic solution	Ratio to coefficients of x_2
$s_1 = 200$	$\frac{200}{2} = 100$
$x_3 = 230$	$\frac{230}{0} \Rightarrow$ ignore
$s_3 = 420$	$\frac{420}{4} = 105$

s_1 leaves the solution. The new tableau is

Basic solution	x_0	x_1	x_2	x_3	s_1	s_2	s_3	Solution
x_0	1	4	0	0	1	2	0	1350
x_2	0	$-1/4$	1	0	$1/2$	$-1/4$	0	100
x_3	0	$3/2$	0	1	0	$1/2$	0	230
s_3	0	2	0	0	-2	0	1	20

This is the optimal solution since all the coefficients in the indicator row are non-negative.

The optimal solution is given by

$x_1 = 0,\ x_2 = 100,\ x_3 = 230,\ \ s_1 = 0,\ s_2 = 0,\ s_3 = 20$ and $x_0 = 1350$

Bibliography

Lucey, T. (2002). *Quantitative Techniques* (6th ed.). Cengage Learning.

Taha, H. A. (2017). *Operation Research An introduction* (10th ed.). Prentice-Hall, Inc.