

OPERATIONS RESEARCH

LECTURE NINE

Duality and Sensitivity Analysis

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INTRODUCTION

This lecture will focus on solving minimization problems in linear programming together with sensitivity analysis.

Intended learning outcomes

At the end of this lecture, you will be able to define the dual of a LPP, Perform sensitivity analysis on LPP and apply sensitivity analysis in interpretation of business problems.

References

These lecture notes should be supplemented with relevant topics from the books listed in the Bibliography at the end of the lecture

Dual and Minimization

A solution of a standard maximum problem produces the solution of an associated standard minimum problem and vice-versa. These associated problems are called the **dual** of each other.

Solving minimization problems with Dual

A standard minimization problem is of the form

Minimize $y_0 = c_1y_1 + c_2y_2 + c_3y_3 + \dots + c_ny_n$

Subject to

1. Find the dual standard maximum problem. This is obtained through

- Form the augmented matrix for the given system of linear inequalities, with the top row consisting of the coefficients of the objective function. That is:

$$\begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_n & \vdots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & \vdots & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & \vdots & b_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & \vdots & b_m \end{bmatrix}$$

- Form the transpose of this matrix

$$\begin{bmatrix} c_1 & \vdots & a_{11} & a_{21} & a_{31} & \vdots & a_{m1} \\ c_2 & \vdots & a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ c_3 & \vdots & a_{13} & a_{23} & a_{33} & \vdots & a_{m3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_n & \vdots & a_{1n} & a_{2n} & a_{3n} & \vdots & a_{mn} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \vdots & b_1 & b_2 & b_3 & \vdots & b_m \end{bmatrix}$$

With the last row being the coefficients of the objective function.

- Develop the dual maximization problem using the transposed matrix

$$\text{Maximize } x_0 = b_1x_1 + b_2x_2 + b_3x_3 + \cdots + b_mx_m$$

Subject to

$$x_1, x_2, x_3, \dots x_m \geq 0$$

2. Solve the dual maximization problem using the simplex method.
3. The maximum value of the dual objective function (x_0), will be the same as the minimum value of the primal objective function (y_0).
4. The optimum value of the primal decision variables will be the entries in the indicator row of the columns corresponding to the slack variables.

Example

Use dual theory to solve the LPP given below

$$\begin{aligned}
 &\text{Minimize } w = 2x_1 + 10x_2 + 8x_3 \\
 &\text{Subject to } \begin{aligned}
 &x_1 + x_2 + x_3 \geq 6 \\
 &x_2 + 2x_3 \geq 8 \\
 &-x_1 + 2x_2 + 2x_3 \geq 4 \\
 &x_1, x_2, x_3 \geq 0
 \end{aligned}
 \end{aligned}$$

Solution

The augmented matrix of the system is $\begin{bmatrix} 2 & 10 & 8 & 0 \\ 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ -1 & 2 & 2 & 4 \end{bmatrix}$

The transpose then becomes $\begin{bmatrix} 2 & 1 & 0 & -1 \\ 10 & 1 & 1 & 2 \\ 8 & 1 & 2 & 2 \\ 0 & 6 & 8 & 4 \end{bmatrix}$

This then gives the dual problem as

$$\begin{aligned}
 &\text{Maximize } z = 6y_1 + 8y_2 + 4y_3 \\
 &\text{Subject to } \begin{aligned}
 &y_1 - y_3 \leq 2 \\
 &y_1 + y_2 + 2y_3 \leq 10 \\
 &y_1 + 2y_2 + 2y_3 \leq 8 \\
 &y_1, y_2, y_3 \geq 0
 \end{aligned}
 \end{aligned}$$

In standard form:

$$\text{Maximize } z - 6y_1 - 8y_2 - 4y_3 = 0$$

$$\begin{aligned} \text{Subject to } & y_1 - y_3 + s_1 = 2 \\ & y_1 + y_2 + 2y_3 + s_2 = 10 \\ & y_1 + 2y_2 + 2y_3 + s_3 = 8 \\ & y_1, y_2, y_3, s_1, s_2, s_3 \geq 0 \end{aligned}$$

The initial tableau in using the simplex procedure is

Basic	z	y_1	y_2	y_3	s_1	s_2	s_3	RHS
z	1	-6	-8	-4	0	0	0	0
s_1	0	1	0	-1	1	0	0	2
s_2	0	1	1	2	0	1	0	10
s_3	0	1	2	2	0	0	1	4

The final simplex tableau is

Basic	z	y_1	y_2	y_3	s_1	s_2	s_3	RHS
z	1	0	0	2	2	0	4	36
y_1	0	1	0	-1	1	0	0	2
s_2	0	0	0	3/2	-1/2	1	-1/2	5
y_2	0	0	1	3/2	-1/2	0	1/2	3

The optimal primal solution from the dual final simplex tableau is:

$$\text{Minimum } w = \text{Maximum } z = 36$$

$x_1 = 2$, $x_2 = 0$, $x_3 = 4$ these are from the coefficients of the slack variables in the indicator row.

Note

1. Slack variables correspond to the primal basic variables in the optimal solution.
2. Values in the indicator row corresponding to the columns of the slack variables gives the optimum values of the basic primal variables.

3. Values of the slack variables of the primal are given by the indicator row under the non-basic variables of the dual.
4. The value of the objective function is the same for both the primal and the dual problems.

Interpretation of the Simplex solution.

Example

A company can produce three products P , Q , and R .

The products yield a contribution of Shs 8, Shs 5 and Shs 10 respectively.

The products use a machine which has 400 hours capacity in the next period. Each unit of the products uses 2, 3 and 1 hour respectively of the machine's capacity.

There are only 150 units available in the period of a special component which is used singly in products P and R .

200Kgs only of a special alloy is available in the period. Product P uses 2Kgs per unit and product R uses 4Kgs per unit.

There is an agreement to produce no more than 50 units of product Q in the period.

Formulate and solve the LOM that would enable the company determine the optimal production plan.

Solution

Let x_i $i = 1, 2, 3$ be the number of units of product P , Q , R

$$\text{Maximize } z = 8x_1 + 5x_2 + 10x_3$$

$$2x_1 + 3x_2 + x_3 \leq 400 \quad (\text{Machine hours constraint})$$

$$x_1 + x_3 \leq 150 \quad (\text{Component constraint})$$

$$\text{Subject to } x_1 + 4x_3 \leq 200 \quad (\text{Alloy constraint})$$

$$x_2 \leq 50 \quad (\text{Sales constraint})$$

$$x_1, x_2, x_3 \geq 0$$

In standard form:

$$\begin{aligned}
 &\text{Maximize } z - 8x_1 - 5x_2 - 10x_3 = 0 \\
 &\quad 2x_1 + 3x_2 + x_3 + s_1 = 400 \quad (\text{Machine hours constraint}) \\
 &\quad x_1 + x_3 + s_2 = 150 \quad (\text{Component constraint}) \\
 \text{Subject to } &\quad x_1 + 4x_3 + s_3 = 200 \quad (\text{Alloy constraint}) \\
 &\quad x_2 + s_4 = 50 \quad (\text{Sales constraint}) \\
 &\quad x_1, x_2, x_3, s_1, s_2, s_3, s_4 \geq 0
 \end{aligned}$$

The slack variables represent the spare capacity in the limitations.

From the simplex method the final tableau is:

Basic	x_0	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
z	1	0	0	6	0	0	4	5	1050
s_1	0	0	0	-3	1	0	-1	-3	50
s_2	0	0	0	-1	0	1	-1/2	0	50
x_1	0	1	0	2	0	0	-1/2	0	100
x_2	0	0	1	0	0	0	0	1	50

Information from the tableau can be interpreted as

1. Reading from the RHS of the final tableau:
 - Optimum product mix $x_1 = 100$ and $x_2 = 50$ means that the company should produce 100 of product P , and 50 of product Q .
 - $s_1 = 50$ and $s_2 = 50$. Note that s_1 corresponds to the machine hours constraint, thus, 50 machine hours remain unused at optimum. Similarly 50 components are unused at optimum.
 - The other two slack variables do not have values on the RHS of the final simplex tableau meaning $s_3 = s_4 = 0$. This means that alloy and sales constraints have no unused capacity at optimum implying the constraints they represent are fully utilized.
 - The maximum contribution at optimum (value of the objective function) is 1050.
 - $x_3 = 0$ implies that product R was not in the optimum production plan.

2. Reading from the indicator row

- The value 6 under the x_3 column means that if any unit of product R was produced, then the overall contribution would fall by sh 6.
- The values for the slack variables in the indicator row are valuations of resources and are known as **shadow prices**.

The shadow price of a binding constraint provides valuable guidance to the management as it indicates the magnitude of change in the overall contribution that would result from altering the amount of the available scarce resource. Therefore:

- $s_1 = s_2 = 0$ means the no value would be gained by increasing both the machine hours and number of components.
- $s_3 = 4$ this means that for every kilogram of alloy available, sh 4 extra overall contribution will be gained.
- $s_4 = 5$ means the every extra unit of Q that was allowed to be produced, the overall contribution would increase by sh 5.

Note – only the constraints s_3 and s_4 are binding constraints (fully utilized) have non-zero shadow prices.

The shadow prices of the binding constraints can be used to confirm the overall contribution i.e.

$$\begin{array}{ll} \text{Alloy availability} &= 200\text{kg} \times 4 = \text{sh } 1800 \\ \text{Sales constraint} &= 50 \times 5 = \text{sh } 250 \\ \text{Total contribution} &= \text{sh } 1050 \end{array}$$

Sensitivity analysis (Post-optimality analysis)

This is usually undertaken to explore the effect of changes in the LP parameters on the optimum solution. It is concerned with the extent of sensitivity of the optimum solution on change in one or more of:

- i. The profit or cost coefficients of the objective function.
- ii. The LHS coefficients of the variables in the constraints.
- iii. The RHS quantities of the constraints.

This analysis can be of the following types.

Type I

To determine the range within which each of the profit (cost) coefficients can be varied without altering the basic variables though the optimum value would actually vary.

Type II

To indicate the extent of increase of the RHS quantities without altering the existing basic variables.

Type III

To investigate the effect of adding or deleting constraints and/or variables on the optimum product mix.

Type IV

To determine the sequence of the basic solutions that become optimum as the changes in the LP parameters are extended further and further.

Bibliography

Lucey, T. (2002). *Quantitative Techniques* (6th ed.). Cengage Learning.

Taha, H. A. (2017). *Operation Research An introduction* (10th ed.). Prentice-Hall, Inc.