OPERATIONS RESEARCH

LECTURE TEN

Transportation problems (1)

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INTRODUCTION

This lecture will focus on the North-West corner rule of the Transportation problem as the first method of solving transportation problems.

Intended learning outcomes

At the end of this lecture, you will be able to differentiate between demand and supply variables, solve a transportation problems using the North-West corner method.

References

These lecture notes should be supplemented with relevant topics from the books listed in the Bibliography at the end of the lecture

Transportation models

This model seeks the determination of a transportation plan of a commodity from a number of sources to a number of destinations with the main objective of minimizing the cost of transportation of the commodity.

The data for this model includes

a. Amount of supply at each source.
b. Amount of demand at each destination.
c. The unit transportation cost of the commodity from each source to each destination.

Let a_i be the number of supply units available at source $i = 1, 2, \dots, m$ and let b_j be the number of demand units required at destination $j = 1, 2, \dots, n$. Let c_{ij} be the per unit transportation cost on route (i, j) joining source i to destination j.

Let $x_{ij} \geq 0$ be the number of units transported from source i to destination j, then the equivalent linear programming model is given by

Minimize $x_0 = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

Subject to
$$\sum_{j=1}^{n} x_{ij} \leq a_i \quad i = 1, 2, \dots, m \text{ (supply constraints)}$$
$$\sum_{i=1}^{m} x_{ij} \leq b_j \quad j = 1, 2, \dots, n \text{ (demand constraints)}$$
$$x_{ij} \geq 0$$

The first set of constraints stipulates that the sum of the commodities from the source cannot exceed its supply. Similarly, the second set of constraints requires that the sum of the commodities must satisfy demand.

When the total supply equals total demand, the resultant formulation is called a **balanced transportation** model.

A balanced transportation model has the two set of constraints equal. This can easily be seen since

$$\sum_{j=1}^{n} b_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} \right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_{ij} \right) = \sum_{i=1}^{m} a_i$$

Suppose $m = 3$ and $n = 5$, then the above linear programming problem can be put in table form as:

From \ To	1	2	3	4	5	a_i (Supply)
1	c_{11} x_{11}	c_{12} x_{12}	c_{13} x_{13}	c_{14} x_{14}	c_{15} x_{15}	a_1
2	c_{21} x_{21}	c_{22} x_{22}	c_{23} x_{23}	c_{24} x_{24}	c_{25} x_{25}	a_2
3	c_{31} x_{31}	c_{32} x_{32}	c_{33} x_{33}	c_{34} x_{34}	c_{35} x_{35}	a_3
b_j (Demand)	b_1	b_2	b_3	b_4	b_5	

Example

Formulate a linear programming model for the following

To / From	w_1	w_2	w_3	w_4	Supply
f_1	21	16	25	13	11
f_2	17	18	14	23	13
f_3	32	27	18	41	19
Demand	6	10	12	15	

The transportation tableau this problem is:

To / From	w_1	w_2	w_3	w_4	Supply
f_1	21 x_{11}	16 x_{12}	25 x_{13}	13 x_{14}	11
f_2	17 x_{21}	18 x_{22}	14 x_{23}	23 x_{24}	13
f_3	32 x_{31}	27 x_{32}	18 x_{33}	41 x_{34}	19
Demand	6	10	12	15	

Which is a balanced transportation model

The linear optimization model then is

Minimize $x_0 = (21x_{11} + 16x_{12} + 25x_{13} + 13x_{14}) + (17x_{21} + 18x_{22} + 14x_{23} + 23x_{24}) + (32x_{31} + 27x_{32} + 18x_{33} + 41x_{34})$

Subject to

$$\left.\begin{aligned} x_{11} + x_{12} + x_{13} + x_{14} &= 11 \\ x_{21} + x_{22} + x_{23} + x_{24} &= 13 \\ x_{31} + x_{32} + x_{33} + x_{34} &= 19 \end{aligned}\right\} \text{Supply constraints}$$

$$\left.\begin{aligned} x_{11} + x_{21} + x_{31} &= 6 \\ x_{12} + x_{22} + x_{32} &= 10 \\ x_{13} + x_{23} + x_{33} &= 12 \\ x_{14} + x_{24} + x_{34} &= 15 \end{aligned}\right\} \text{Demand constraints}$$

$$x_{ij} \geq 0 \text{ for all } i = 1,2,3; \quad j = 1,2,3,4$$

Basic steps of the transportation technique

Step 1 – determine the starting feasible solution

Step 2 – determine an entering variable from among the non-basic variables. If all such variables satisfy the optimality condition of the simplex method, then stop. Otherwise, go to the next step.

Step 3 – determine a leaving variable using the feasibility condition from among the variables of the current basic solutions. Find the new basic solution and return to step 2.

There are three main methods of finding the initial starting feasible solution to transportation problems namely:

1. North-west corner rule
2. Lowest cost entry method
3. Vogel's approximation method.

1. North-west corner rule

The procedure of finding the initial feasible solution is

- Select the north-west corner cell of the transportation tableau and allocate as many units as possible equal to the minimum between available supply and demand requirements.
- Adjust the supply and demand numbers in the respective rows and columns allocations.
- If the supply for the first row is exhausted, then move to the first cell in the second row and first column and allocate the maximum units possible. Alternatively, if the demand for the first column is satisfied, then move horizontally to the next cell in the second column and first row and allocate the maximum possible units.
- If for any cell supply equals demand, then the next allocation can be made in a cell either in the next row or column.
- Repeat the process until the total available quantity is fully allocated to the cells required.

This gives an initial basic feasible solution if there are $m + n - 1$ independent constraints. Where $m -$ number of rows and $n -$ number of columns.

Example

Consider the following transportation problem.

To / From	1	2	3	4	Supply
1	2	3	1.5	2.5	900
2	4	2.5	2.5	3	750
Demand	300	450	550	350	

Find the starting solution using North-west corner rule

Solution

To / From	1	2	3	4	Supply
1	2 | 300	3 | 450	1.5 | 150	2.5 |	900
2	4 |	2.5 |	2.5 | 400	3 | 350	750
Demand	300	450	550	350	

The starting feasible solution is

$$x_{11} = 300,\ \ x_{12} = 450,\ \ x_{13} = 150,\ \ x_{23} = 400,\ \ x_{24} = 350\ \ - \text{Basic variables}$$
$$x_{14} = \ x_{21} = \ \ x_{22} = \ \ 0\ - \ \ \text{Non} - \text{basic variables}$$

The initial transportation cost therefore is

$$x_0 = 2(300) + 3(450) + 1.5(150) + 2.5(0) + 4(0) + 2.5(0) + 2.5(400) + 3(350) = 4225$$

From the starting feasible solution, we determine the optimum solution using the **stepping stone** method.

Stepping stone method.

After determining the starting feasible solution, we determine the entering and leaving variable by constructing a closed loop for the current entering variable.

Let us define a closed path as a loop that starts and ends at the designated non-basic variable. It consists of successive horizontal and vertical connected segments whose end points must be a non-basic variable. This means that every corner element of the loop must contain a basic variable.

Cell evaluation to determine the entering variable is then done by calculating of the net cost change that can be obtained by introduction of any of the non-basic variables into the solution. A negative net cost change indicates that a cost reduction can be obtained by making the change, and a positive one indicates a cost increase.

The entering variable will then correspond to the cell with the most negative net cost change.

The leaving variable is selected from among the current basic variables (corner elements) of the loop that would decrease when the entering variable increases. This will be a variable smallest variable (allocation) corresponding to a negative corner of the loop.

The optimum solution is obtained when all the cell evaluations are non-negative.

WORKED EXAMPLE IN THE CLASS RECORDING.

<u>Bibliography</u>

Lucey, T. (2002). *Quantitative Techniques* (6th ed.). Cengage Learning.

Taha, H. A. (2017). *Operation Research An introduction* (10th ed.). Prentice-Hall, Inc.