OPERATIONS RESEARCH

LECTURE TWELVE

Assignment models

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INTRODUCTION

This lecture will focus on the second special linear optimization model known as the Assignment models and use of the Hungarian method to solve the assignment problems.

Intended learning outcomes

At the end of this lecture, you will be able to identify an assignment problem and apply the Hungarian method to solve assignment problems.

References

These lecture notes should be supplemented with relevant topics from the books listed in the Bibliography at the end of the lecture

Assignment models

These models deals in allocation of various resources (items) to various activities (receivers) on a one to one basis in such a way that the resultant effectiveness is optimized. For example, assignment of available sales-force to different regions, vehicles to routes, products to factories, contracts to bidders, machines to jobs, development engineers to several construction sites and so on.

Consider a situation of assigning m jobs (or workers) to n machines. A job $i = 1, 2, \dots,$ m when assigned to machine $j = 1, 2, \dots, n$ is assumed to incur a cost c_{ij}. The objective is to assign the jobs (one job per machine) at the least total cost.

The general assignment model table is as below:

$$A = \quad Jobs \begin{matrix} & Machines \\ & \begin{matrix} 1 & 2 & 3 & \dots & n \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ \vdots \\ m \end{matrix} & \begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2n} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ c_{m1} & c_{m2} & c_{m3} & .. & c_{mn} \end{pmatrix} \end{matrix}$$

Note

The assignment model table must always be a square matrix. This is achieved by always adding dummy rows or columns with zero elements where there is no balance.

If a job cannot be assigned to a certain machine, the corresponding cost (c_{ij}) is taken to be a very high cost.

Mathematically, the assignment model is

Let $x_{ij} = \begin{cases} 0 & \text{if the j}^{\text{th}} \text{ job is not assigned to the i}^{\text{th}} \text{ machine} \\ 1 & \text{if the j}^{\text{th}} \text{job is assigned to the i}^{\text{th}} \text{ machine} \end{cases}$

Then

Maximize $x_0 = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

Subject to
$$\sum_{i=1}^{m} x_{ij} = 1 \quad j = 1, \ 2, \ \dots, \ n$$
$$\sum_{j=1}^{n} x_{ij} = 1 \quad i = 1, \ 2, \ \dots, \ m$$
$$x_{ij} = 1 \ or \ 0$$

Example

Obtain the linear programming model for the following assignment problem

$$\text{child} \begin{matrix} & \text{chore} \\ & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1 & 4 & 6 & 3 \\ 9 & 7 & 10 & 9 \\ 4 & 5 & 11 & 7 \\ 8 & 7 & 8 & 5 \end{pmatrix} \end{matrix}$$

Solution

$$\text{Minimize } x_0 = \begin{matrix} x_{11} + 4x_{12} + 6x_{13} + 3x_{14} + \\ 9x_{21} + 7x_{22} + 10x_{23} + 9x_{24} + \\ 4x_{31} + 5x_{32} + 11x_{33} + 7x_{34} + \\ 8x_{41} + 7x_{42} + 8x_{43} + 5x_{44} \end{matrix}$$

$$\text{Subject to} \quad \begin{matrix} x_{11} + x_{12} + x_{13} + x_{14} = 1 \\ x_{21} + x_{22} + x_{23} + x_{24} = 1 \\ x_{31} + x_{32} + x_{33} + x_{34} = 1 \\ x_{41} + x_{42} + x_{43} + x_{44} = 1 \\ x_{11} + x_{21} + x_{31} + x_{41} = 1 \\ x_{12} + x_{22} + x_{32} + x_{42} = 1 \\ x_{13} + x_{23} + x_{33} + x_{42} = 1 \\ x_{14} + x_{24} + x_{34} + x_{44} = 1 \\ x_{ij} = 1 \ or \ \ 0 \qquad i, j = 1,2,3,4 \end{matrix}$$

The Hungarian method

This method is used to find the optimal solution to the assignment problem.

The steps to be followed are:

1. Extract the original cost square matrix. If the matrix is not square add a dummy row/column.
2. Identify the minimum element in each row and subtract it from all the elements in the row.
3. From the matrix resulting from (2) above, identify the minimum element in each column and subtract it from all the elements in the column.

   Note:

   Step (1) and (2) ensures that there exist at least one zero in each row and column.
4. Identify the optimum solution as the feasible assignment associated with the zero elements of the matrix obtained in (2) and (3)
5. If no feasible assignment can be secured from the above procedure:
   a. Draw the minimum number of vertical and horizontal lines in the last reduced matrix that cover all the zero entries.
   b. Select the smallest element in the reduced matrix that does not have any line through it. Add this element to all elements that occur at the intersection of

two lines and subtract it from all elements that do not have any line through them. The other elements of the matrix remain the same, then repeat $5(a)$.

c. If the number of lines drawn equals the order of the matrix, this is this is the optimum table. If not, repeat $5(b)$.

d. Using the optimum table, the actual assignments can be made by:
   - Assigning to any zero unique to both a column and row.
   - Assigning to any zero unique to a column or a row.
   - Ignoring assignments already made and repeating the second assignment until all assignments are made.

Example 1

Find the optimal assignment that will minimize cost for the assignment below.

$$\begin{array}{cc} & \textit{Contractor} \\ & \begin{array}{cccc} A & B & C & D \end{array} \\ \textit{Assembly} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 16 & 14 & 15 & 18 \\ 12 & 13 & 16 & 14 \\ 14 & 13 & 11 & 12 \\ 16 & 18 & 15 & 17 \end{pmatrix} \end{array}$$

Solution

Extracting the cost matrix, we have

$$\begin{pmatrix} 16 & 14 & 15 & 18 \\ 12 & 13 & 16 & 14 \\ 14 & 13 & 11 & 12 \\ 16 & 18 & 15 & 17 \end{pmatrix}$$

For the rows we have;

$$\begin{pmatrix} 2 & 0 & 1 & 4 \\ 0 & 1 & 4 & 2 \\ 3 & 2 & 0 & 1 \\ 1 & 3 & 0 & 2 \end{pmatrix}$$

For the columns we have;

$$\begin{pmatrix} 2 & 0 & 1 & 3 \\ 0 & 1 & 4 & 1 \\ 3 & 2 & 0 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix}$$

Assign starting with the row or column that has only one zero, then follow the rest at random making one allocation per row or column.

$$\begin{pmatrix} 2 & \boxed{0} & 1 & 3 \\ \boxed{0} & 1 & 4 & 1 \\ 3 & 2 & 0 & \boxed{0} \\ 1 & 3 & \boxed{0} & 1 \end{pmatrix}$$

Assembly	Contractor
1	B
2	A
3	D
4	C

Example 2

At the end of the cycle of schedules a tracking firm has a surplus of one vehicle in cities A, B, C, D, E, and F. The distance between cities with the deficit are shown in the table below. Find the assignment of the surplus vehicle to the deficit cities that will result to the minimum total distance.

$$\begin{array}{ccc} & & To \\ & & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ From\ cities & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 13 & 11 & 16 & 23 & 19 & 9 \\ 11 & 19 & 26 & 16 & 17 & 13 \\ 12 & 11 & 4 & 9 & 6 & 10 \\ 7 & 15 & 9 & 14 & 14 & 13 \\ 9 & 13 & 12 & 8 & 14 & 11 \end{pmatrix} \end{array}$$

Solution

There exists more requirements than means. We add a dummy row and the cost or distance associated with the row is zero. Adding a dummy row to the above distance matrix extracted makes it a 6×6 matrix.

Steps

Extract the 6×6 matrix

$$\begin{pmatrix} 13 & 11 & 16 & 23 & 19 & 9 \\ 11 & 19 & 26 & 16 & 17 & 13 \\ 12 & 11 & 4 & 9 & 6 & 10 \\ 7 & 15 & 9 & 14 & 14 & 13 \\ 9 & 13 & 12 & 8 & 14 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For the row we will have

$$\begin{pmatrix} 4 & 2 & 7 & 14 & 10 & 0 \\ 0 & 8 & 15 & 5 & 6 & 2 \\ 8 & 7 & 0 & 5 & 2 & 6 \\ 0 & 8 & 2 & 7 & 7 & 6 \\ 1 & 5 & 4 & 0 & 6 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For the columns, the matrix will remain the same because of the dummy row. With the vertical and horizontal lines drawn, an assignment with the zero total cannot be made therefore the additional computations to produce more zeros are needed.

The smallest element without a line through it in the reduced matrix is 2, therefore:

$$\begin{pmatrix} 4 & 0 & 7 & 14 & 8 & 0 \\ 0 & 6 & 15 & 5 & 4 & 2 \\ 8 & 5 & 0 & 5 & 0 & 6 \\ 0 & 6 & 2 & 7 & 5 & 6 \\ 1 & 3 & 4 & 0 & 4 & 3 \\ 2 & 0 & 2 & 2 & 0 & 2 \end{pmatrix}$$

The lines drawn are not equal to the order of the matrix, therefore not optimal and the procedure is repeated as below with 2 being the smallest element once again

$$\begin{pmatrix} 6 & 0 & 7 & 16 & 8 & 0 \\ 0 & 4 & 13 & 5 & 2 & 0 \\ 10 & 5 & 0 & 7 & 0 & 6 \\ 0 & 4 & 0 & 7 & 3 & 4 \\ 1 & 1 & 2 & 0 & 2 & 2 \\ 4 & 0 & 2 & 4 & 0 & 2 \end{pmatrix}$$

The number of lines drawn equal the order of the matrix, hence optimal matrix.

Start assigning with the row or column that has only one zero, then follow the rest at random, making one allocation per row or column.

$$\begin{pmatrix} 6 & 0 & 7 & 16 & 8 & \boxed{0} \\ \boxed{0} & 4 & 13 & 5 & 2 & 0 \\ 10 & 5 & 0 & 7 & \boxed{0} & 6 \\ 0 & 4 & \boxed{0} & 7 & 3 & 4 \\ 1 & 1 & 2 & \boxed{0} & 2 & 2 \\ 4 & \boxed{0} & 2 & 4 & 0 & 2 \end{pmatrix}$$

From	To	Mileage
1	F	9
2	A	11
3	E	6
4	C	9
5	D	8
	Total	43

Bibliography

Lucey, T. (2002). *Quantitative Techniques* (6th ed.). Cengage Learning.

Taha, H. A. (2017). *Operation Research An introduction* (10th ed.). Prentice-Hall, Inc.