# OPERATIONS RESEARCH

# LECTURE SIX

# Linear programming (2)


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**INTRODUCTION**

This lecture will focus on graphical solution to linear programming problems.

**Intended learning outcomes**

At the end of this lecture, you will be able to Solve LLP using graphical method

**References**

These lecture notes should be supplemented with relevant topics from the books listed in the Bibliography at the end of the lecture

**Solving mathematical models for linear programming**

There are two main methods of solving linear programming problems. In this lecture, we will learn the first method which is graphical solution to the problems.

This method is used to solve a linear programming problem with only two decision variables.

**Procedure**

1. Formulate the linear optimization model.
2. Graph the feasible space.
3. Determine the coordinates of each of the corner points within the feasible space.
4. Find the value of the objective function at each of the corner points.
5. For a bounded region, the solution is given by the corner point producing the optimum value of the objective function.
6. For an unbounded region, check that the solution exists. If it does, it will occur at a corner point.

**Example**

A manufacturer produces two different cloth products Silk and Cotton. Silk has the contribution of three dollars per unit, and Cotton four dollars per unit. The manufacturer wishes to establish the weekly production plan to optimize the contribution. The production data are as follows:

| | Per unit | | |
|---|---|---|---|
| | Machine (hours) | Labour (hours) | Material (Kg) |
| Silk | 4 | 4 | 1 |
| Cotton | 2 | 6 | 1 |
| Total availability/week | 100 | 180 | 40 |

Because of a trade agreement, sales of Silk are limited to a weekly maximum of 20 units and to honour an agreement with an old established customer at least 10 units of Cotton must be sold per week. Formulate and solve the linear programming program.

**Solution**

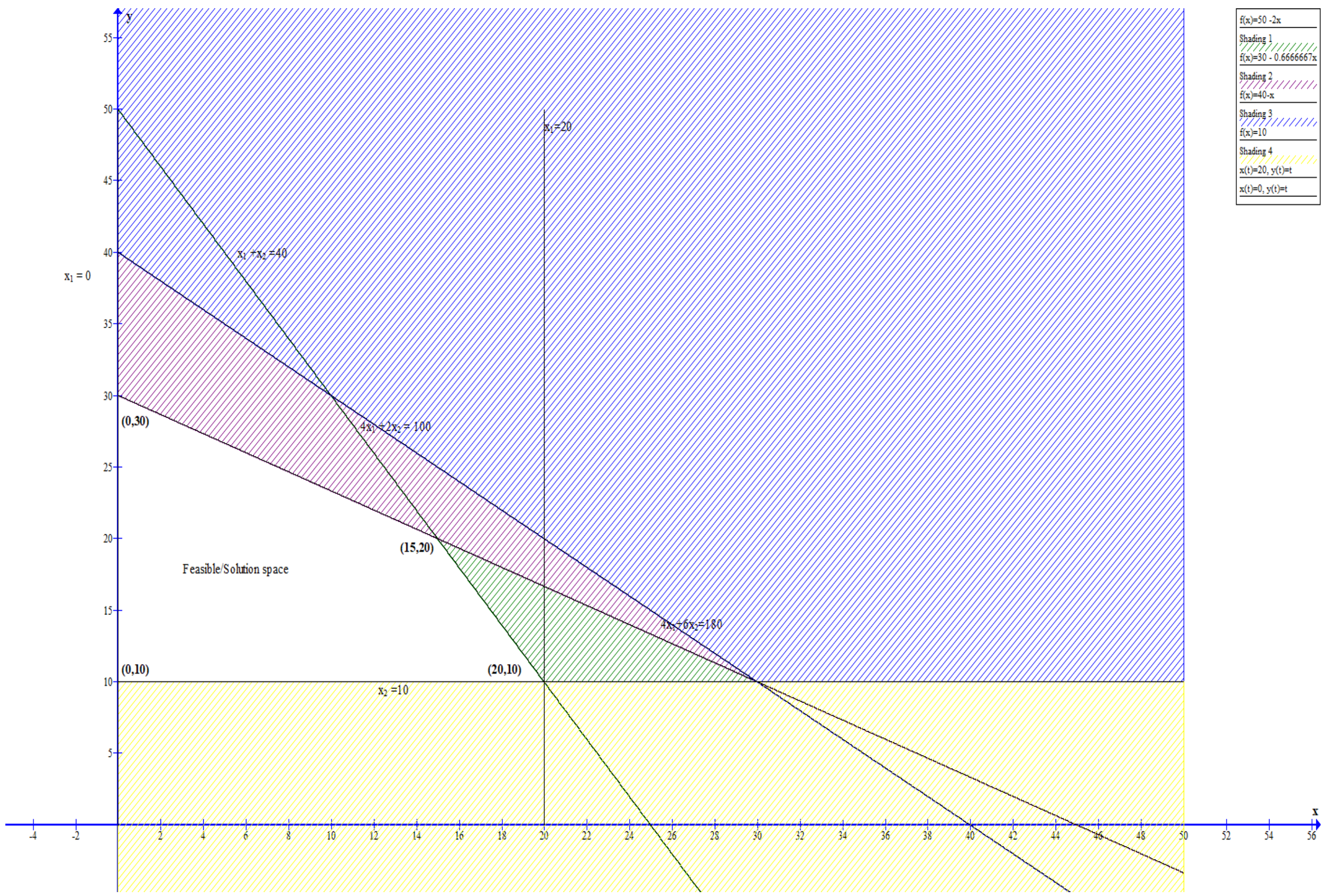
Let $x_i, \quad i = 1, 2$ be the units of Silk and Cotton to be produced respectively, then the optimization model will be:

Maximize $x_0 = 3x_1 + 4x_2$

$$\text{Subject to} \quad \begin{array}{c} 4x_1 + 2x_2 \leq 100 \\ 4x_1 + 6x_2 \leq 180 \\ x_1 + x_2 \leq 40 \\ x_1 \leq 20 \\ x_2 \geq 10 \\ x_1 \geq 0 \end{array}$$

Since this problem has only two decision variables, we will use graphical method to find the solution.

We plot all the constraints on a single cartesian plane, with inequality signs changed into equal signs. The direction in which each constraint holds is then determined by observing the direction of the inequality sign. We then shade the unwanted region. The area which is satisfied by all the constraints including all the boundaries is known as the solution space or the feasible space.




We are interested in determining the points in the solution space which yields the maximum value of $x_0$. This is determined by identifying the coordinates of the corner points within the feasible space i.e., $(0,10), (0,30), (20,10)$ and $(15,20)$, and then determining the corresponding value of the objective function at each of the corner points. Thus;

| $x_1$ | $x_2$ | Objective function $x_0$ |
|---|---|---|
| 0 | 10 | 40 |
| 0 | 30 | 120 |
| 20 | 10 | 100 |
| 15 | 20 | 125 |

The coordinates $(15, 20)$ represent the optimal value of the objective function $x_0 = 125$.

**Effects of constraints**

Constraints are the conditions which limit the decision variables to their feasible values.
A constraint is binding if altering it also alters the optimal solution.
When using graphical solution method, binding constraints (scarce resources) will pass through the optimum solution point. In the above example, the binding constraints are

$$4x_1 + 2x_2 \leq 100 \text{ (machime hours)}$$
$$4x_1 + 6x_2 \leq 180 \text{ (labour hours)}$$

Less severe constraints that do not affect the optimum solution are known as non-binding constraints (abundant resources). These constraints do not pass through the optimum solution point in graphical method. In the above example, the non-binding constraint is

$$x_1 + x_2 \leq 40 \ \left(\text{materials (kg)}\right)$$

The management should be given information about the value of the scarce resources. These valuations are known as the shadow/dual prices. They are obtained from the amount of increase/decrease in in the optimal contribution that will arise if 1 more/less unit of the scarce resource was made available.

**Determining the shadow prices**

There are two methods of finding the shadow prices when dealing with graphical solution to linear optimization models:

a. **Arithmetic method**

The process here is to determine the magnitude of change on the optimal solution when one constraint is increased by a single unit as the other is held constant and vice versa.
That is, assume that one more machine hour is available, but the labour hours remain constant at 180. The resulting difference in the optimal contribution will be the shadow price corresponding to the machine hour.

Hence,

$$4x_1 + 2x_2 = 101$$

$$4x_1 + 6x_2 = 180$$

Solving the two equations simultaneously, we obtain

$$x_2 = 19.75 \qquad x_1 = 15.35$$

The corresponding value of the objective function then becomes

$$x_0 = 3x_1 + 4x_2 = 3(15.375) + 4(19.75) = 125.125$$

The change in the optimal contribution is

$$125.125 - 125 = 0.125$$

Therefore, the shadow price per machine hour is 0.125 dollars.

Similarly,

$$4x_1 + 2x_2 = 100$$

$$4x_1 + 6x_2 = 181$$

Solving the two equations simultaneously, we obtain

$$x_2 = 20.25 \qquad x_1 = 14.875$$

The corresponding value of the objective function then becomes

$$x_0 = 3x_1 + 4x_2 = 3(14.875) + 4(20.25) = 125.625$$

The change in the optimal contribution is

$$125.625 - 125 = 0.625$$

Therefore, the shadow price per machine hour is 0.625 dollars.

**b. Dual formulation**

This involves formulating a dual optimization model for a primal sub-model of the binding constraints, then solving for the dual variables.

From the above example,

The primal sub-model of the binding constraints is:

$$\text{Maximize } x_0 = 3x_1 + 4x_2$$

$$\text{Subject to } \begin{matrix} 4x_1 + 2x_2 \leq 100 \\ 4x_1 + 6x_2 \leq 180 \end{matrix}$$

$$x_1 \geq 0, \;\; x_2 \geq 0$$

The dual problem will be

$$\text{Minimize } y_0 = 100y_1 + 180y_2$$

$$\text{Subject to } \begin{matrix} 4y_1 + 4y_2 \geq 3 \\ 2y_1 + 6y_2 \geq 4 \end{matrix}$$

$$y_1 \geq 0, \;\; y_2 \geq 0$$

Solving the two constraint equations simultaneously results give the values of the dual variables as

$$y_1 = 0.125, \qquad y_2 = 0.625$$

Which are the shadow prices corresponding to the machine hours and labour hours respectively

The contribution of the binding constraint to the optimal solution can be confirmed by multiplying the available units by the contribution per unit (shadow prices).

Since the available machine hours was $100$, and each machine hour contribute $0.125$ dollars, then the total contribution to the optimal solution from machine hours is

$$100(0.125) = 12.5$$

Similarly, the total contribution from the labour hour having aa maximum availability of 180 will be:

$$180(0.625) = 112.5$$

Therefore, the total contribution to the optimal solution from the binding constraints is

$$12.5 + 112.5 = 125$$

**Bibliography**


Lucey, T. (2002). *Quantitative Techniques* (6th ed.). Cengage Learning.

Taha, H. A. (2017). *Operation Research An introduction* (10th ed.). Prentice-Hall, Inc.