Beyond Nash: Domination, Rationalization and Correlation CONT'D

Theorem *(Bernheim/Pearce (1984))*
The set of rationalizable strategies is nonempty and contains at least one pure strategy for each player. Further, each $\sigma_i \in R_i$ is (in Σ_i) a best response to an element of $\times_{j \neq i} Conv(R_j)$.

Comparing the constructions of undominated strategies with rationalizable strategies, we note that

$$\Sigma_i^0 = \Sigma_i, \quad \text{and} \quad \tilde{\Sigma}_i^0 = \Sigma_i.$$

In the nth iteration, the undominated strategies are constructed as

$$\Sigma_i^n = \Big\{ \sigma_i \in \Sigma_i^{n-1} \mid \forall_{\sigma_i' \in \Sigma_i^{n-1}} \exists_{\sigma_{-i} \in \times_{j \neq i} \mathrm{Conv}(\Sigma_j^{n-1})} u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \Big\},$$

where as rationalizable strategies are constructed as

$$\tilde{\Sigma}_i^n = \Big\{ \sigma_i \in \tilde{\Sigma}_i^{n-1} \mid \exists_{\sigma_{-i} \in \times_{j \neq i} \mathrm{Conv}(\tilde{\Sigma}_j^{n-1})} \forall_{\sigma_i' \in \tilde{\Sigma}_i^{n-1}} u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \Big\}.$$

Finally,

$$\Sigma_i^\infty = \bigcap_{n=0}^{\infty} \Sigma_i^n, \Sigma^\infty = \times_i \Sigma_i^\infty, \quad \text{and} \quad R_i = \bigcap_{n=0}^{\infty} \tilde{\Sigma}_i^n, R = \times_i R_i.$$

A direct examination of these constructions reveals that $\tilde{\Sigma}_i^n \subseteq \Sigma_i^n$ and hence, $R \subseteq \Sigma^\infty$. Also, note that the undominated strategies are computing the minmax values where as the rationalizable strategies compute maxmin values.

Correlated Equilibrium

Aumann's Example

	L	R
U	5,1	0,0
D	4,4	1,5

There are 3 Nash equilibria:

- A pure strategy: (U, L) $\mapsto$ Pay-off = 5,1,
- A pure strategy: (D, R) $\mapsto$ Pay-off = 1,5, and
- A mixed strategy: $((1/2, 1/2),\ (1/2, 1/2)) \mapsto$ Pay-off = (2.5, 2.5).

Suppose that there is a publicly observable random variable with $Pr(H) = Pr(T) = 1/2$. Let the players play (U, L) if the outcome is H, and (D, R) if the outcome is T. Then the pay-off is (3, 3).

By using publicly observable random variables, the players can obtain any pay-off vector in the convex hull of the set of Nash equilibria pay-offs.

Players can improve (without any prior contracts) if they can build a device that sends different but correlated signals to each of them.

Formal Definitions

- "Expanded Games" with a correlating device.
- Nash equilibrium for the expanded game.

Definition *Correlating device is a triple*

$$(\Omega, \{H_i\}_{\mathcal{I}}, p)$$

- *Ω = a (finite) state space corresponding to the outcomes of the device.*

- $p =$ *probability measure on the state space* Ω
- $H_i =$ *Information Partition for player* i.

  Assigns an $h_i(\omega)$ *to each* $\omega \in \Omega$ *such that* $\omega \in h_i(\omega)$.

$$h_i \,:\, \Omega \rightarrow H_i \,:\, \omega \mapsto h_i(\omega).$$

Player i's posterior belief about Ω *are given by Bayes' law:*

$$\forall_{\omega \in h_i}\; p(\omega | h_i) = \frac{p(\omega)}{p(h_i)}.$$

Pure Strategies for the Expanded Game

Given a correlating device $(\Omega, \{H_i\}, p)$, we can define strategies for the expanded game as follows: Consider a map

$$\tau_i \,:\, \Omega \rightarrow S_i \,:\, \omega \mapsto \tau_i(\omega),$$

such that $\tau_i(\omega) = \tau_i(\omega')$, if $\omega' \in h_i(\omega)$.

The strategies are *adapted* to the information structure.

Definition **(1)** *A* correlated equilibrium τ *relative to information structure* $(\Omega, \{H_i\}, p)$ *is a Nash equilibrium in strategies that are adapted to information structure. That is,* $(\tau_1, \tau_2, \ldots, \tau_I)$ *is a correlated equilibrium if*

$$\forall_i\, \forall_{\tilde{\tau}_i} \sum_{\omega \in \Omega} p(\omega) u_i(\tau_i(\omega), \tau_{-i}(\omega)) \geq \sum_{\omega \in \Omega} p(\omega) u_i(\tilde{\tau}_i(\omega), \tau_{-i}(\omega)).$$

Using the Bayes' rule, an equivalent condition would be:

$$\begin{aligned} &\forall_i\, \forall_{h_i \in H_i, p(h_i) > 0}\, \forall_{s_i \in S_i} \\ &\qquad \sum_{\omega | h_i(\omega) = h_i} p(\omega | h_i) u_i(\tau_i(\omega), \tau_{-i}(\omega)) \\ &\geq \sum_{\omega | h_i(\omega) = h_i} p(\omega | h_i) u_i(s_i, \tau_{-i}(\omega)). \end{aligned}$$

Correlated Equilibrium and Universal Device

"Universal Device" that signals each player how that player should play.

Definition **(2)** *A* correlated equilibrium *is any probability distribution $p(.)$ over the pure strategies $S_1 \times S_2 \times \cdots \times S_I$ such that, for every player i, and every function $d(i) : S_i \to S_i$*

$$\sum_{s \in S} p(s) u_i(s_i, s_{-i}) \geq \sum_{s \in S} p(s) u_i(d(s_i), s_{-i}).$$

Using the Bayes' rule, an equivalent condition would be:

$$\begin{aligned} &\forall_i \ \forall_{s_i \in S_i, p(s_i)>0} \ \forall_{s_i' \in S_i} \\ &\qquad \sum_{s_{-i} \in S_{-i}} p(s_{-i}|s_i) u_i(s_i, s_{-i}) \\ &\geq \sum_{s_{-i} \in S_{-i}} p(s_{-i}|s_i) u_i(s_i', s_{-i}). \end{aligned}$$

Equivalence of correlated equilibria under Def(1) and Def(2):
Claim:
Def(1) $\Leftarrow$ Def(2):
Choose $\Omega = S$. $h_i(s) = \{s' | s_i' = s_i\}$. Leave $p(s)$ unchanged.

Claim:
Def(1) $\Rightarrow$ Def(2):
Let τ be an equilibrium w.r.t. $(\Omega, \{H_i\}, \tilde{p})$. Define

$$p(s) = \sum \{\tilde{p}(\omega) | \tau_1(\omega) = s_1, \ldots, \tau_I(\omega) = s_I, \omega \in \Omega\}.$$

Let

$$J_i(s_i) = \{\omega | \tau_i(\omega) = s_i\}.$$

Thus

$$\tilde{p}(J_i(s_i)) = p(s_i) = \text{ probability that player } i \text{ is told to play } s_i.$$

$$\sum_{\omega \in J_i(s_i)} \frac{\tilde{p}(\omega)}{\tilde{p}(J_i(s_i))} \tau_{-i}(\omega).$$

It is the mixed strategy of the rivals that player i believes he faces, conditional on being told to play s_i, and it is a convex combination of the distributions conditional on each h_i such that $\tau_i(h_i) = s_i$.