Learning a la Milgrom and Roberts

Adaptive Learning and Undominated Sets

Example: Battle of Sexes

W\M	Ballet(B)	Football(F)
Ballet(B)	2,1	0,0
Football(F)	0,0	1,2

Let $\{x(t)\}$ be a sequence of strategy profiles. We show that $x(t) = (F, B)$ is consistent with sophisticated learning.

$$\forall_{\hat{t}}\ \{x(s)|\hat{t} \leq s < t\} = \{(F, B)\}.$$

Thus, we have

$$\begin{aligned} F_W^{\epsilon,0}(\hat{t}, t) &= U_W^{\epsilon}\Big(\{(F, B)\}\Big) = B \\ F_M^{\epsilon,0}(\hat{t}, t) &= U_M^{\epsilon}\Big(\{(F, B)\}\Big) = F \end{aligned}$$

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Thus

$$F^{\epsilon,0}(\hat{t},t) = \{\ (B,F)\ \}.$$

Similarly,

$$F^{\epsilon,1}(\hat{t},t) = U^{\epsilon}\Big(\{(B,F),(F,B)\}\Big) = \{B,F\} \times \{B,F\}.$$

Continuing in this fashion, we get

$$F^{\epsilon,\infty}(\hat{t},t) = \{B,F\} \times \{B,F\}.$$

Thus

$$x(t+1) = (F,B) \in F^{\epsilon,\infty}(\hat{t},t),$$

is consistent with sophisticated learning.

Convergence

Definition *A sequence of strategy profiles $\{x(t)\}$ converges omitting correlation to a correlated strategy profile*

$$G \in \Delta(S)$$

if (1) and (2) hold:

1. *G_n^t converges weakly to the marginal distribution G_n for all n.*
2. $$\forall_{\epsilon>0}\ \exists_{\bar{t}}\ \forall_{t\geq\bar{t}}\ \forall_{n\in N}\ d[x_n(t), supp(G_n)] < \epsilon,$$

   Define $d[x,T] \equiv \inf_{y\in T} \|x-y\|$.

The sequence converges to the correlated strategy $G \in \Delta(S)$ if in addition

G^t converges weakly to G.

Definition *A sequence $\{x(t)\}$ converges omitting correlation to a mixed strategy Nash equilibrium if*

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1. *It replicates the empirical frequency of the separate mixed strategies and*

2. *It eventually plays only pure strategies that are in or near the support of the equilibrium mixed strategies.*

Theorem *If $\{x(t)\}$ converges omitting correlation to a correlated equilibrium in the game Γ, then $\{x(t)\}$ is consistent with adaptive learning.*

Proof Sketch:

G^t converges to a correlated equilibrium G.

$\Rightarrow$ G_n consists of best responses to G_{-n}

$\Rightarrow$ For sufficiently large t, $x_n(t)$ is within ϵ of G_n

$\Rightarrow$ Since S_n is compact and π is continuous

$$\forall_{y_n \in G_n} \exists_{z_{-n} \in G_{-n}} \exists_{\delta > 0} \; \pi_n(x_n(t), z_{-n}) + \delta \geq \pi_n(y_n, z_{-n})$$

$\Rightarrow$

$$x_n(t) \in U_n^{\delta}\Big(\{x(s) \mid \hat{t} \leq s < t\}\Big).$$

Theorem *Suppose that the sequence $\{x(t)\}$ is consistent with adaptive learning and that it converges to x^*. Then x^* is a pure strategy Nash equilibrium.*

Proof Sketch*:*

Assume that x^* is not a Nash equilibrium

$\Rightarrow$ $\exists_{n \in N} \forall_{\epsilon > 0} \{x_n^*\} \neq U^{\epsilon}(\{x^*\})$.

$\Rightarrow$ Player n must play $x_n' \neq x_n^*$ i.o.

$\Rightarrow$ $x_n(t)$ does not converge to x_n^*

$\Rightarrow$ **Contradiction**.

Theorem *Let $\{x(t)\}$ be consistent with sophisticated learning. Then for each $\epsilon > 0$ and $k \in \mathbb{N}$ there exists a time $t_{\epsilon k}$ after which (i.e., for $t \geq t_{\epsilon k}$)*

$$x(t) \in U^{\epsilon k}(S).$$

Proof Sketch:

Fix $\epsilon > 0$. Define $t_k \equiv t_{\epsilon k}$ (Change in notation).

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Case $k = 0$: $t_0 = 0$. $x(t) \in U^{\epsilon}(S)$.

Case $k = j + 1$: By the inductive hypothesis there exists a t_j such that

$$\forall_{t \geq t_j}\ x(t) \in U^{\epsilon j}(S).$$

Hence

$$\{x(s) \mid t_j \leq s < t\} \subseteq U^{\epsilon j}(S).$$

Since $\{x(t)$ is consistent with sophisticated learning, we can choose

$$\hat{t} = t_j, \quad t_{j+1} = \max(\hat{t}, \bar{t}).$$

Then

$$\forall_{t \geq t_{j+1}}\ x(t) \in F^{\epsilon\infty}(t_j, t).$$

Claim:

$$F^{\epsilon\infty}(t_j, t) \subseteq U^{\epsilon, j+1}(S).$$

Equivalently,

$$\forall_i\ F^{\epsilon i}(t_j, t) \subseteq U^{\epsilon, j+1}(S).$$

It then follows that

$$\begin{aligned} F^{\epsilon 0}(t_j, t) &= U^{\epsilon}\Big(\{x(s) \mid t_j \leq s < t\}\Big) \\ &\subseteq U^{\epsilon}\Big(U^{\epsilon, j}(S)\Big) \;=\; U^{\epsilon, j+1}(S). \end{aligned}$$

$$\begin{aligned} F^{\epsilon, i+1}(t_j, t) &= U^{\epsilon}\Big(F^{\epsilon, i}(t_j, t) \;\cup\; \{x(s) \mid t_j \leq s < t\}\Big) \\ &\subseteq U^{\epsilon}(U^{\epsilon, j+1}(S) \cup U^{\epsilon, j}(S)) \\ &= U^{\epsilon}(U^{\epsilon, j}(S)) \;=\; U^{\epsilon, j+1}(S). \end{aligned}$$

$$\bigcap_{k} \bigcap_{\epsilon > 0} U^{\epsilon k}(S) = \bigcap_{k} U^{0k}(S) = U^{0\infty}(S).$$

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