Portfolios and Markets

Portfolio Theory

Itô Calculus

X = asset price at time t. In a continuous time model, one can study the return on the asset dX/X over a small period of time dt.

$$\frac{dX}{X} = \mu\ dt + \sigma\ dZ.$$

This is a so-called Itô process.

$$\begin{aligned} \mu &= \text{average rate of growth: } \textbf{DRIFT} \\ \sigma &= \text{volatility: } \textbf{DIFFUSION} \end{aligned}$$

Market Model

Assume that there are m stocks, represented by m Itô processes:

$$X_1(t), X_2(2), \ldots, X_m(t).$$

Furthermore,

$$\frac{dX_i}{X_i} = \mu_i\ dt + \sum_{j=1}^{m} \sigma_{ij}\ dZ_j,$$

Here, Z_j's are independent Brownian motions.

$$\mu = \textbf{ Drift Vector } = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{pmatrix}.$$

$$\sigma = \textbf{ Diffusion Matrix } = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mm} \end{pmatrix}$$

$$\Sigma = \textbf{ Instantaneous Covariance Matrix } = \sigma\, \sigma^T.$$

In general, the term dZ corresponds to a *Wiener Process.*

- dZ = Normal Random Variable.
- $dZ \sim \mathcal{N}(0, \sqrt{dt})$. Mean of dZ is zero and variance of dZ is dt.

$$dZ = \phi\sqrt{dt}, \qquad E[\phi] = 0, \qquad E[\phi^2] = 1.$$

This holds in continuous time in the limit as $dt \to 0$.

Lemma **Itô's Lemma** [Analogous to Taylor's theorem in case of functions of random variables. The key ideas is based on the observation that with probability 1, $dZ^2 \to dt$ as $dt \to 0$.]

Suppose $f(X)$ is a function of X (where X is possibly stochastic).

$$\begin{aligned} df &= \frac{\partial f}{\partial X} dX + \frac{1}{2}\frac{\partial^2 f}{\partial X^2} dX^2 + \textit{ smaller order terms} \\ dX^2 &= (\mu X\, dt + \sigma X dZ)^2 \\ &= \sigma^2 X^2 dZ^2 + 2\sigma\mu X^2 dZ\, dt + \mu^2 X^2 dt^2 \\ &\to \sigma^2 X^2 dt \qquad \textit{as } dt \to 0 \\ df &= \frac{\partial f}{\partial X}(\mu X dt + \sigma X dZ) + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 f}{\partial X^2} dt \\ &= \left(\mu X \frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 f}{\partial X^2}\right) dt + \sigma X \frac{\partial f}{\partial X} dZ. \end{aligned}$$

Example

$$\frac{dX}{X} = \mu \ dt + \sigma \ dZ$$

Let $f(X) = \ln X$. Then

$$\frac{\partial f}{\partial X} = \frac{1}{X}, \quad \& \quad \frac{\partial^2 f}{\partial X^2} = -\frac{1}{X^2}$$

$$\begin{aligned}
df &= \frac{\partial f}{\partial X} dX + \frac{1}{2}\frac{\partial^2 f}{\partial X^2} dX^2 \\
&= \frac{dX}{X} - \frac{1}{2X^2}\sigma^2 X^2 dt \\
&= \frac{dX}{X} - \frac{\sigma^2}{2} dt \\
d(\ln X) &= \frac{dX}{X} - \frac{\sigma^2}{2} dt \\
\frac{dX}{X} &= d(\ln X) + \frac{\sigma^2}{2} dt \\
\int_0^t \frac{dX}{X} &= \int_0^t d(\ln X) + \frac{1}{2}\int_0^t \sigma^2 dt \\
&= \ln X(t) - \ln X(0) + \frac{1}{2}\int_0^t \sigma^2 dt \\
\exp\left\{\int_0^t \frac{dX}{X}\right\} &= \frac{X(t)}{X(0)} \exp\left\{\frac{1}{2}\int_0^t \sigma^2 dt\right\}
\end{aligned}$$

Rebalanced Portfolio

Market Model with m stocks:

$$\begin{aligned}
\frac{dX_i(t)}{X_i(t)} &= \mu_i(t)dt + \sum_{j=1}^{m} \sigma_{ij}(t) dZ_j(t) \\
\Sigma(t) &= \sigma(t)\sigma(t)^T.
\end{aligned}$$

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A portfolio of long stocks at time t is identified by its weighted vector process $b(t) \in B$.

$$B = \left\{ b \in \mathbb{R}^m \mid b_i \geq 0, \sum_{i=1}^{m} b_i = 1 \right\}.$$

Rebalanced Portfolio
(A self-financing portfolio without dividends).

$$\begin{aligned} \frac{dS(t)}{S(t)} &= \sum_{i=1}^{m} b_i(t) \frac{dX_i(t)}{X_i(t)} \\ &= \left(\sum_{i=1}^{m} b_i(t)\mu_i(t) \right) dt + \left(\sum_{i=1}^{m} \sum_{j=1}^{m} b_i(t)\sigma_{ij}(t) dZ_j \right). \end{aligned}$$

Let $g(S) = \ln S$ and $f(X) = \sum b_i \ln X_i = \ln \prod X_i^{b_i}$.

$$\begin{aligned} dg &= \frac{dS}{S} - \frac{1}{2S^2}(b^T \Sigma b) S^2 dt \\ \frac{dS}{S} &= d(\ln S) + \frac{1}{2}(b^T \Sigma b) dt \\ df &= \sum b_i \frac{dX_i}{X_i} - \sum \frac{1}{2X_i^2}(b_i \Sigma_{ii}) X_i^2 dt \\ \sum b_i \frac{dX_i}{X_i} &= d(\sum b_i \ln X_i) + \frac{1}{2} \sum b_i \Sigma_{ii} dt \end{aligned}$$

Hence

$$\begin{aligned} d(\ln S) &= d(\sum b_i \ln X_i) - \frac{1}{2} b^T \Sigma b dt + \frac{1}{2} \sum b_i \Sigma_{ii} dt \\ \ln \frac{S(t;b)}{S(0)} &= \sum b_i \ln \frac{X_i(t)}{X_i(0)} - \frac{1}{2} b^T \Lambda b + \frac{1}{2} \sum b_i \Lambda_{ii}, \end{aligned}$$

where $\Lambda \equiv \int_0^t \Sigma(s) ds$.

$$S(t;b) = S(0) \prod_{i=1}^{m} \left(\frac{X_i(t)}{X_i(0)} \right)^{b_i} \exp\left\{ -\frac{1}{2} b^T \Lambda b + \frac{1}{2} \sum \Lambda_{ii} b_i \right\}.$$

Maximizing the above expression we have

$$S^*(t) = \max_{b \in B} S(t; b) = S(t; b^*(t))$$

Note that $b^*(t)$ = optimal solution of the following quadratic programming problem:

$$\max_{b \in B} -\frac{1}{2} b^T \Lambda b + \sum_{i=1}^{m} \left(\ln \frac{X_i(t)}{X_i(0)} + \frac{1}{2} \Lambda_{ii} \right) b_i.$$

Define the matrix V, an $(m-1) \times (m-1)$ symmetric positive semidefinite matrix

$$V = (V_{ij}) \quad V_{ij} = \Lambda_{ij} - \Lambda_{im} - \Lambda_{jm} + \Lambda_{mm}, \;\; 1 \leq i, j \leq m.$$

Lemma *If V = positive definite then the portfolio problem has a unique optimal solution.*

Definition *A stochastic process $X(t)$ is* weakly regular *if*

$$\forall_t \;\; |E[X(t)]| < \infty.$$

$$\lim \frac{E[X(t)]}{t} = \theta \qquad \textit{exists}$$

$$\frac{X(t)}{t} \to \theta \qquad \textit{in probability as } t \to \infty.$$

The stock market model is weakly regular (easily satisfied if the market is stationary)

$$\forall_t \;\; |E[\Lambda(t)]| < \infty, \quad \& \quad |E[\ln X(t)]| < \infty,$$

$$\lim \frac{E[\Lambda(t)]}{t} = \Sigma^\infty, \quad \& \quad \lim \frac{E[\ln X(t)]}{t} = \eta^\infty \qquad \text{exist}$$

$$\frac{\Lambda(t)}{t} \to \Sigma^\infty, \quad \& \quad \frac{\ln X(t)}{t} \to \eta^\infty \qquad \text{in probability as } t \to \infty.$$

Note that

$$\begin{aligned}
\frac{dX_i}{X_i} &= \mu_i dt + \sum \sigma_{ij} dZ_j \\
d(\ln X_i) &= \frac{dX_i}{X_i} - \frac{dX_i^2}{2X_i^2} \\
&= \left(\mu_i - \frac{\Sigma_{ii}}{2}\right) dt + \sum \sigma_{ij} dZ_j \\
\eta_i^\infty &= \mu_i^\infty - \frac{\Sigma_{ii}^\infty}{2}.
\end{aligned}$$

Thus

$$\mu_i^\infty = \eta_i^\infty + \frac{\Sigma_{ii}^\infty}{2}.$$

Similarly,

$$\begin{aligned}
\frac{dS}{S} &= \sum b_i \mu_i dt + \sum \sum b_i \sigma_{ij} dZ_j \\
d(\ln S) &= \left(b^T \mu - \frac{1}{2} b^T \Sigma b\right) dt + \sum \sum b_i \sigma_{ij} dZ_j \\
r(b) &= \lim \frac{E[\ln S(t;b)]}{t} = -\frac{1}{2} b^T \Sigma^\infty b + b^T \mu^\infty.
\end{aligned}$$

Asymptotically optimal constant weight $b^\infty \in B$.

$$r(b^\infty) = \max_{b \in B} r(b) = \max_{b \in B} -\frac{1}{2} b^T \Sigma^\infty b + b^T \mu^\infty.$$

Optimal Portfolio

Recall

$$S(t;b) = S(0) \prod_{i=1}^m \left(\frac{X_i(t)}{X_i(0)}\right)^{b_i} \exp\left\{-\frac{1}{2} b^T \Lambda b + \frac{1}{2} \sum \Lambda_{ii} b_i\right\}.$$

$$V_{ij}(t) = \Lambda_{ij} - \Lambda_{im} - \Lambda_{jm} + \Lambda_{mm}.$$

Define

$$\lambda_i = \ln\left(\frac{X_m(t)}{X_m(0)}\right) - \ln\left(\frac{X_i(t)}{X_i(0)}\right) - \frac{V_{ii}(t)}{2}.$$

Notation:

$$b = (b', b_m) \quad b'_1 + \cdots + b'_{m-1} + b_m = 1, \quad b'_i \geq 0, b_m > 0.$$

Rewriting the previous equation, we have

$$S(t;b) = S(0)\frac{X_m(t)}{X_m(0)} \exp\left\{-\frac{1}{2}b'^T V b' - \lambda^T b'\right\}.$$

The above value $S(t;b)$ is maximized at $b' = \beta^*$

$$\begin{aligned} V(t)\beta^*(t) &= -\lambda(t) \\ \beta^*(t) &= -V^{-1}(t)\lambda(t) \end{aligned}$$

$$S^*(t) = S(0)\frac{X_m(t)}{X_m(0)} e^{\beta^{*T} V \beta^*/2},$$

and

$$S(t;b) = S^*(t) \ \exp\left\{-\frac{1}{2}(b' - \beta^*)^T V (b' - \beta^*)\right\}.$$

$$\frac{S(t;b)}{S^*(t)} = \exp\left\{-\frac{1}{2}(b' - \beta^*)^T V (b' - \beta^*)\right\}.$$

Long Term Effects

$$\left.\begin{aligned} V_{ij} &= \Lambda_{ij} - \Lambda_{im} - \Lambda_{jm} + \Lambda_{mm} \\ J^{\infty}_{ij} &= \Sigma^{\infty}_{ij} - \Sigma^{\infty}_{im} - \Sigma^{\infty}_{jm} + \Sigma^{\infty}_{mm} \end{aligned}\right\} \quad \lim \frac{V(t)}{t} = J^{\infty}$$

$$\left.\begin{aligned} \lambda_i &= \ln\left(\tfrac{X_m(t)}{X_m(0)}\right) - \ln\left(\tfrac{X_i(t)}{X_i(0)}\right) - \tfrac{V_{ii}(t)}{2} \\ \gamma^{\infty}_i &= \eta^{\infty}_m - \eta^{\infty}_i - \tfrac{J^{\infty}_{ii}}{2} \\ &= \mu^{\infty}_m - \tfrac{\Sigma^{\infty}_{mm}}{2} - \mu^{\infty}_i - \tfrac{\Sigma^{\infty}_{ii}}{2} \\ &\quad - \tfrac{\Sigma^{\infty}_{ii}}{2} + \Sigma^{\infty}_{im} - \tfrac{\Sigma^{\infty}_{mm}}{2} \\ &= \mu^{\infty}_m - \mu^{\infty}_i + \Sigma^{\infty}_{im}. \end{aligned}\right\} \quad \lim \frac{\lambda_i(t)}{t} = \gamma^{\infty}_i$$

Since

$$r(b) = -\frac{1}{2}b^T\Sigma^\infty b + b^T\mu^\infty = -\frac{1}{2}b'^T J^\infty b' - b'^T\gamma^\infty,$$

it is maximized at

$$\beta^\infty = -(J^\infty)^{-1}\gamma^\infty.$$

Note, however, that

$$\beta^*(t) = -\left(\frac{V(t)}{t}\right)^{-1}\left(\frac{\lambda(t)}{t}\right) \to -(J^\infty)^{-1}\gamma^\infty = \beta^\infty.$$

Problem: Construction of b^∞ requires the long-term average of future instantaneous expected returns and covariances. This however is impossible.
Remedy: *Universal Portfolio*

Universal Portfolio

Rebalanced portfolio with weights:

$$\hat{b}_i(t) = \frac{\int_B b_i S(t;b)db}{\int_B S(t;b)db}.$$

Let

$$\bar{S} = \frac{\int_B S(t;b)db}{\int_B db}$$

Note that

$$\bar{S}(0) = \hat{S}(0).$$

Furthermore,

$$\frac{d\bar{S}}{\bar{S}} = \frac{\int_B dS(t;b)db}{\int_B S(t;b)db} = \frac{\int_B \sum_i S(t;b)b_i(dX_i/X_i)db}{\int_B S(t;b)db} = \sum_i \hat{b}_i(t)\frac{dX_i}{X_i} = \frac{d\hat{S}}{\hat{S}}$$

Hence,

$$\forall_t \, \hat{S}(t) = \bar{S}(t).$$

Lemma *The wealth accumulated by a universal portfolio is given by*

$$\hat{S}(t) = \frac{\int_B S(t;b)db}{\int_B db}.$$

This is the average wealth accumulated by all possible portfolios.

Competitiveness

$$S(t;b) = S^*(t)e^{-\frac{1}{2}(b'-\beta^*)^T V(b'-\beta^*)}.$$

Let $x = V^{1/2}(t)(b' - b'^*)$. Thus

$$\Delta(t) = V^{1/2}(t)(B' - b'^*),$$

where

$$B' = \left\{ b' \in \mathbb{R}^{m-1} \mid b'_i \geq 0, \sum b'_i < 1 \right\}.$$

Note that

$$\text{Vol}\,(B') = \frac{1}{(m-1)!}.$$

We have

$$\hat{S}(t) = \frac{S^*(t) \int_{\Delta(t)} e^{-|x|^2/2}\, dx}{|V(t)|^{1/2}(1/(m-1)!)}.$$

$$\begin{aligned}
\frac{\hat{S}(t)}{S^*(t)} &= \frac{(m-1)! \int_\Delta e^{-|x|^2/2} dx}{\left(\left|\frac{V(t)}{t}\right|\right)^{1/2} t^{m-1/2}} \\
&= \frac{(m-1)!(\sqrt{2\pi})^{m-1}}{|J^\infty|^{1/2} t^{m-1/2}} \\
&= \frac{(m-1)!}{|J^\infty|^{1/2}} \left(\frac{2\pi}{t}\right)^{m-1/2}.
\end{aligned}$$

Thus,

$$\frac{1}{t}\ln\frac{\hat{S}(t)}{S^*(t)} = \frac{C(m) - C'(m)\ln t}{t} \rightarrow 0$$

and

$$\frac{\ln \hat{S}(t)}{t} \rightarrow \frac{\ln S^*(t)}{t} \rightarrow \frac{\ln S(t; b^\infty)}{t}.$$