LECTURE 11: APPLICATIONS OF NON LINEAR EQUATIONS

As we know that the solution of the exponential model for the population growth is

$$\boxed{P(t) = P_0\, e^{kt}}$$

P_0 being the initial population. From this solution we conclude that

(a) If $k > 0$ the population grows and expand to infinity i.e. $\boxed{\lim_{t\to\infty} P(t) = +\infty}$

(b) If $k < 0$ the population will shrink to approach 0, which means extinction.

Note that:

(1) The prediction in the first case ($k > 0$) differs substantially from what is actually observed, population growth is eventually limited by some factor!

(2) Detrimental effects on the environment such as pollution and excessive and competitive demands for food and fuel etc. can have inhibitive effects on the population growth.

Logistic equation:

Another model was proposed to remedy this flaw in the exponential model. This is called the **logistic model** (also called **Verhulst-Pearl model**).

Suppose that $a > 0$ is constant average rate of birth and that the death rate is proportional to the population $P(t)$ at any time t. Thus if $\frac{1}{P}\frac{dP}{dt}$ is the rate of growth per individual then

$$\boxed{\frac{1}{P}\frac{dP}{dt} = a - bP \quad \text{or} \quad \frac{dP}{dt} = P(a - bP)}$$

where b is constant of proportionality. The term $\boxed{-bP^2,\ b > 0}$ can be interpreted as inhibition term. When $b = 0$, the equation reduces to the one in exponential model. Solution to the logistic equation is also very important in ecological, sociological and even in managerial sciences.

Solution of the Logistic equation:

The logistic equation

$$\boxed{\frac{dP}{dt} = P(a - bP)}$$

can be easily identified as a **nonlinear** equation that is separable. The constant solutions of the equation are given by

$$\boxed{P(a - bP) = 0}$$

$$\Rightarrow \ P = 0 \quad \text{and} \quad P = \frac{a}{b}$$

For non-constant solutions we separate the variables

$$\frac{dP}{P(a - bP)} = dt$$

Resolving into partial fractions we have

$$\left[\frac{1/a}{P} + \frac{b/a}{a - bP}\right] dP = dt$$

Integrating $$\frac{1}{a}\ln|P| - \frac{1}{a}\ln|a - bP| = t + C$$

$$\ln\left|\frac{P}{a - bP}\right| = at + aC$$

or $$\frac{P}{a - bP} = C_1 e^{at} \quad \text{where} \quad C_1 = e^{aC}$$

Easy algebraic manipulations give

$$P(t) = \frac{aC_1 e^{at}}{1 + bC_1 e^{at}} = \frac{aC_1}{bC_1 + e^{-at}}$$

Here C_1 is an arbitrary constant. If we are given the initial condition $P(0) = P_0$, $P_0 \neq \frac{a}{b}$ we obtain $C_1 = \frac{P_0}{a - bP_0}$. Substituting this value in the last equation and simplifying, we obtain

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}$$

Clearly $$\lim_{t \to \infty} P(t) = \frac{aP_0}{bP_0} = \frac{a}{b},$$ **limited growth**

Note that $P = \frac{a}{b}$ is a **singular solution** of the logistic equation.

Special Cases of Logistic Equation:

1. Epidemic Spread

Suppose that one person infected from a contagious disease is introduced in a fixed population of n people.

The natural assumption is that the rate $\frac{dx}{dt}$ of spread of disease is proportional to the number $x(t)$ of the infected people and number $y(t)$ of people not infected people. Then

$$\frac{dx}{dt} = kxy$$

Since $x + y = n + 1$

Therefore, we have the following initial value problem

$$\frac{dx}{dt} = kx(n+1-x), \quad x(0) = 1$$

The last equation is a **special case of the logistic equation** and has also been used for the **spread of information** and the **impact of advertising** in centers of population.

2. A Modification of LE:

A modification of the nonlinear logistic differential equation is the following

$$\frac{dP}{dt} = P(a - b\ln P)$$

has been used in the studies of solid tumors, in actuarial predictions, and in the growth of revenue from the sale of a commercial product in addition to growth or decline of population.

Example:

Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not infected, determine the number of infected students after 6 days if it is further observed that after 4 days $x(4) = 50$.

Solution

Assume that no one leaves the campus throughout the duration of the disease. We must solve the initial-value problem

$$\frac{dx}{dt} = kx(1000 - x), \quad x(0) = 1 \quad .$$

We identify

$$a = 1000k \quad \text{and} \quad \mathrm{b} = \mathrm{k}$$

Since the solution of logistic equation is

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}$$

Therefore we have

$$x(t) = \frac{1000k}{k + 999ke^{-1000kt}} = \frac{1000}{1 + 999e^{-1000kt}}.$$

Now, using x(4)= 50, we determine k

$$50 = \frac{1000}{1 + 999e^{-4000k}}$$

We find

$$k = \frac{-1}{4000} \ln \frac{19}{999} = 0.0009906.$$

Thus

$$x(t) = \frac{1000}{1 + 999e^{-0.9906\,t}}$$

Finally

$$x(6) = \frac{1000}{1 + 999e^{-5.9436}} = 276 \text{ students} \quad .$$

Chemical reactions:

In a first order chemical reaction, the molecules of a substance A decompose into smaller molecules. This decomposition takes place at a rate proportional to the amount of the first substance that has not undergone conversion. The disintegration of a radioactive substance is an example of the first order reaction. If X is the remaining amount of the substance A at any time t then

$$\frac{dX}{dt} = k\,X$$

$k < 0$ because X is decreasing.

In a 2nd order reaction two chemicals A and B react to form another chemical C at a rate proportional to the product of the remaining concentrations of the two chemicals.

If X denotes the amount of the chemical C that has formed at time t. Then the instantaneous amounts of the first two chemicals A and B not converted to the chemical C are $\alpha - X$ and $\beta - X$, respectively. Hence the rate of formation of chemical C is given by

$$\frac{dX}{dt} = k\left(\alpha - \mathrm{X}\right)\left(\beta - X\right)$$

where k is constant of proportionality.

Example:

A compound $\boldsymbol{C}$ is formed when two chemicals A and B are combined. The resulting reaction between the two chemicals is such that for each gram of A, 4 grams of B are used. It is observed that 30 grams of the compound C are formed in 10 minutes. Determine the amount of C at any time if the rate of reaction is proportional to the amounts of A and B remaining and if initially there are 50 grams of A and 32 grams of B. How much of the compound C is present at 15 minutes? Interpret the solution as $t \to \infty$

Solution:

If $X(t)$ denote the number of grams of chemical $\boldsymbol{C}$ present at any time t. Then

$$X(0) = 0 \text{ and } X(10) = 30$$

Suppose that there are 2 grams of the compound $\boldsymbol{C}$ and we have used a grams of A and b grams of B then

$$a + b = 2 \quad \text{and} \quad b = 4a$$

Solving the two equations we have

$$a = \frac{2}{5} = 2(1/5) \quad \text{and} \quad b = \frac{8}{5} = 2(4/5)$$

In general, if there were for X grams of C then we must have

$$a = \frac{X}{5} \quad \text{and} \quad b = \frac{4}{5}X$$

Therefore the amounts of A and B remaining at any time t are then

$$50 - \frac{X}{5} \text{ and } 32 - \frac{4}{5}X$$

respectively .

Therefore, the rate at which chemical C is formed satisfies the differential equation

$$\boxed{\frac{dX}{dt} = \lambda\left(50 - \frac{X}{5}\right)\left(32 - \frac{4}{5}X\right)}$$

or

$$\boxed{\frac{dX}{dt} = k(250 - X)(40 - X), \quad k = 4\lambda/25}$$

We now solve this differential equation.

By separation of variables and partial fraction, we can write

$$\boxed{\frac{dX}{(250 - X)(40 - X)} = kdt}$$

$$\boxed{-\frac{1/210}{250 - X}dX + \frac{1/210}{40 - X}dX = kdt}$$

$$\boxed{\ln\left|\frac{250 - X}{40 - X}\right| = 210kt + c_1}$$

$$\boxed{\frac{250 - X}{40 - X} = c_2 e^{210kt} \quad Where\ c_2 = e^{c_1}}$$

When $t = 0$, $X = 0$, so it follows at this point that $c_2 = 25/4$. Using $X = 30$ at $t = 10$, we find

$$\boxed{210k = \frac{1}{10}\ln\frac{88}{25} = 0.1258}$$

With this information we solve for X:

$$\boxed{X(t) = 1000\left(\frac{1 - e^{-0.1258t}}{25 - 4e^{-0.1258t}}\right)}$$

It is clear that as $e^{-0.1258t} \to 0$ as $t \to \infty$. Therefore $X \to 40$ as $t \to \infty$. This fact can also be verified from the following table that $X \to 40$ as $t \to \infty$.

t	10	15	20	25	30	35
X	30	34.78	37.25	38.54	39.22	39.59

This means that there are 40 grams of compound C formed, leaving

$$50-\frac{1}{5}(40)=42 \quad \text{grams of chemical } A$$

and

$$32-\frac{4}{5}(40)=0 \quad \text{grams of chemical } B$$

Miscellaneous Applications

☐ The velocity v of a falling mass m, subjected to air resistance proportional to instantaneous velocity, is given by the differential equation

$$m\frac{dv}{dx}=mg-kv$$

Here $k>0$ is constant of proportionality.

☐ The rate at which a drug disseminates into bloodstream is governed by the differential equation

☐

$$\frac{dx}{dt}=A-Bx$$

Here $A,\ B$ are positive constants and $x(t)$ describes the concentration of drug in the bloodstream at any time t.

☐ The rate of memorization of a subject is given by

$$\frac{dA}{dt}=k_1(M-A)-k_2A$$

Here $k_1>0,\ k_2>0$ and $A(t)$ is the amount of material memorized in time t, M is the total amount to be memorized and $M-A$ is the amount remaining to be memorized.