MTH 401 – DIFFERENTIAL EQUATIONS

LECTURE 5: INTEGRATING FACTOR – TECHNIQUE

If the equation

$$M(x,y)dx + N(x,y)dy = 0$$

is not exact, then we must have

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Therefore, we look for a function ***u (x, y)*** such that the equation

$$u(x,y)M(x,y)dx + u(x,y)N(x,y)dy = 0$$

becomes exact. The function ***u (x, y)*** (if it exists) is called the **integrating factor (IF)** and it satisfies the equation due to the condition of exactness.

$$\frac{\partial M}{\partial y}u + \frac{\partial u}{\partial y}M = \frac{\partial N}{\partial x}u + \frac{\partial u}{\partial x}N$$

This is a partial differential equation and is very difficult to solve. Consequently, the determination of the integrating factor is extremely difficult except for some special cases:

Example

Show that $1/(x^2+y^2)$ is an integrating factor for the equation $(x^2+y^2-x)dx - ydy = 0,$

and then solve the equation.

Solution: Since $M = x^2 + y^2 - x, \quad \mathrm{N} = -y$

Therefore $\frac{\partial M}{\partial y} = 2y, \; \frac{\partial N}{\partial x} = 0$

So that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

and the equation is not exact. However, if the equation is multiplied by $1/(x^2+y^2)$ then the equation becomes

$$\left(1 - \frac{x}{x^2+y^2}\right)dx - \frac{y}{x^2+y^2}dy = 0$$

Now $\quad M = 1 - \dfrac{x}{x^2+y^2}$ and $N = -\dfrac{y}{x^2+y^2}$

Therefore $\quad \dfrac{\partial M}{\partial y} = \dfrac{2xy}{(x^2+y^2)^2} = \dfrac{\partial N}{\partial x}$

So that this new equation is exact. The equation can be solved. However, it is simpler to observe that the given equation can also written

$$dx - \frac{xdx + ydy}{x^2+y^2} = 0 \quad \text{or} \quad dx - \frac{1}{2}d\left[\ln(x^2+y^2)\right] = 0$$

or

$$d\left[x - \frac{\ln(x^2+y^2)}{2}\right] = 0$$

Hence, by integration, we have

$$x - \ln\sqrt{x^2+y^2} = k$$

Case 1:

When $\exists$ an integrating factor *u* (*x*), a function of x only. This happens if the expression

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$$

is a function of x only.

Then the integrating factor $u(x,y)$ is given by

$$u = \exp\left(\int \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} dx\right)$$

Case 2:

When $\exists$ an integrating factor $u(y)$, a function of y only. This happens if the expression

$$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$$

is a function of y only. Then **IF** $u(x,y)$ is given by

$$u = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} dy\right)$$

Case 3:

If the given equation is homogeneous and

$$xM + yN \neq 0$$

Then $$u = \frac{1}{xM + yN}$$

Case 4:

If the given equation is of the form

$$yf(xy)dx + xg(xy)dy = 0$$

and $$xM - yN \neq 0$$

Then $$u = \frac{1}{xM - yN}$$

Once the **IF** is found, we multiply the old equation by u to get a new one, which is exact. Solve the exact equation and write the solution.

Advice: If possible, we should **check** whether or not the new equation is **exact?**

Summary:

Step 1. Write the given equation in the form

$$M(x, y)dx + N(x, y)dy = 0$$

provided the equation is not already in this form and determine M and N.

Step 2. Check for exactness of the equation by finding whether or not

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Step 3. (a) If the equation is not exact, then evaluate

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

If this expression is a function of x only, then

$$u(x) = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} dx\right)$$

Otherwise, evaluate

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}$$

If this expression is a function of y only, then

$$u(y)=\exp\left(\int \frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}dy\right)$$

In the **absence** of these **2 possibilities**, better use some other technique. However, we could also try cases 3 and 4 in step 4 and 5

Step 4. Test whether the equation is homogeneous and

$$xM+yN\neq 0$$

If yes then

$$u=\frac{1}{xM+yN}$$

Step 5. Test whether the equation is of the form

$$yf(xy)dx+xg(xy)dy=0$$

and whether

$$xM-yN\neq 0$$

If yes then

$$u=\frac{1}{xM-yN}$$

Step 6. Multiply old equation by u. if possible, check whether or not the new equation is exact?

Step 7. Solve the new equation using steps described in the previous section.

Illustration:

Example 1

Solve the differential equation

$$\frac{dy}{dx}=-\frac{3xy+y^2}{x^2+xy}$$

Solution:

1. The given differential equation can be written in form

$$(3xy+y^2)dx+(x^2+xy)dy=0$$

Therefore

$$M(x,y)=3xy+y^2$$

$$N(x,y) = x^2 + xy$$

2. Now
$$\frac{\partial M}{\partial y} = 3x + 2y,\ \frac{\partial N}{\partial x} = 2x + y.$$

$$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

3. To find an **IF** we evaluate

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{1}{x}$$

which is a function of x only.

4. Therefore, an **IF** $u(x)$ exists and is given by

$$u(x) = e^{\int \frac{1}{x} dx} = e^{\ln(x)} = x$$

5. Multiplying the given equation with the **IF,** we obtain

$$(3x^2 y + xy^2)dx + (x^3 + x^2 y)dy = 0$$

which is exact. (Please check**!**)

6. This step consists of solving this last exact differential equation.

Solution of new exact equation:

1. Since $\frac{\partial M}{\partial y} = 3x^2 + 2xy = \frac{\partial N}{\partial x}$, the equation is exact.

2. We find $F(x, y)$ by solving the system

$$\begin{cases} \frac{\partial F}{\partial x} = 3x^2 y + xy^2 \\ \frac{\partial F}{\partial y} = x^3 + x^2 y. \end{cases}$$

3. We integrate the first equation to get

$$F(x,y) = x^3 y + \frac{x^2}{2} y^2 + \theta(y)$$

4. We differentiate F w. r. t. **'y'** and use the second equation of the system in step 2 to obtain

$$\frac{\partial F}{\partial y} = x^3 + x^2 y + \theta'(y) = x^3 + x^2 y$$

$\Rightarrow \theta' = 0$, **No dependence on x.**

5. Integrating the last equation to obtain $\theta = C$. Therefore, the function $F(x, y)$ is

$$F(x,y) = x^3 y + \frac{x^2}{2} y^2$$

We don't have to keep the constant C, see next step.

6. All the solutions are given by the implicit equation $F(x, y) = C$ **i.e.**

$$x^3 y + \frac{x^2 y^2}{2} = C$$

Note that it can be verified that the function

$$u(x,y) = \frac{1}{2xy(2x+y)}$$

is another integrating factor for the same equation as the new equation

$$\frac{1}{2xy(2x+y)}(3xy + y^2)dx + \frac{1}{2xy(2x+y)}(x^2 + xy)dy = 0$$

is exact. This means that we may **not have uniqueness** of the integrating factor.

Example 2. Solve

$$\left(x^2 - 2x + 2y^2\right)dx + 2xydy = 0$$

Solution:

$$M = x^2 - 2x + 2y^2$$
$$N = 2xy$$

$$\frac{\partial M}{\partial y} = 4y, \frac{\partial N}{\partial x} = 2y$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The equation is not exact.

Here $$\frac{M_y - N_x}{N} = \frac{4y - 2y}{2xy} = \frac{1}{x}$$

Therefore, I.F. is given by

$$u = \exp\left(\int \frac{1}{x} dx\right)$$

$$u = x$$

$\therefore$ I.F is x.

Multiplying the equation by x, we have

$$\left(x^3 - 2x^2 + 2xy^2\right)dx + 2x^2 ydy = 0$$

This equation is exact. The required Solution is

$$\frac{x^4}{4} - \frac{2x^3}{3} + x^2 y^2 = c_0$$

$$3x^4 - 8x^3 + 12x^2 y^2 = c$$

Example 3

Solve $$dx + \left(\frac{x}{y} - \sin y\right)dy = 0$$

Solution: Here

$$M = 1, \quad N = \frac{x}{y} - \sin y$$
$$\frac{\partial M}{\partial y} = 0, \quad \frac{\partial N}{\partial x} = \frac{1}{y}$$
$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The equation is not exact.

Now

$$\frac{N_x - M_y}{M} = \frac{\frac{1}{y} - 0}{1} = \frac{1}{y}$$

Therefore, the IF is $$u(y) = \exp\int \frac{dy}{y} = y$$

Multiplying the equation by *y*, we have

$$ydx + (x - y\sin y)dy = 0$$

or $$ydx + xdy - y\sin ydy = 0$$

or $$d(xy) - y\sin ydy = 0$$

Integrating, we have

$$xy + y\cos y - \sin y = c$$

Which is the required solution.

Example 4

Solve $$(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0$$

Solution: Comparing with

$$Mdx+Ndy=0$$

we see that

$$M=x^2y-2xy^2 \quad \text{and} \quad N=-(x^3-3x^2y)$$

Since both M and N are homogeneous. Therefore, the given equation is homogeneous.
Now

$$xM+yN=x^3y-2x^2y^2-x^3y+3x^2y^2=x^2y^2\neq 0$$

Hence, the factor u is given by

$$u=\frac{1}{x^2y^2} \qquad \because u=\frac{1}{xM+yN}$$

Multiplying the given equation with the integrating factor u, we obtain.

$$\left(\frac{1}{y}-\frac{2}{x}\right)dx-\left(\frac{x}{y^2}-\frac{3}{y}\right)dy=0$$

Now

$$M=\frac{1}{y}-\frac{2}{x} \quad \text{and} \quad N=\frac{-x}{y^2}+\frac{3}{y}$$

and therefore

$$\frac{\partial M}{\partial y}=-\frac{1}{y^2}=\frac{\partial N}{\partial x}$$

Therefore, the new equation is exact and solution of this new equation is given by

$$\frac{x}{y}-2\ln|x|+3\ln|y|=C$$

Example 5

Solve $$y(xy+2x^2y^2)dx+x(xy-x^2y^2)dy=0$$

Solution:
The given equation is of the form

$$yf(xy)dx+xg(xy)dy=0$$

Now comparing with

$$Mdx + Ndy = 0$$

We see that

$$M = y\left(xy + 2x^2y^2\right) \text{ and } N = x\left(xy - x^2y^2\right)$$

Further

$$xM - yN = x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3$$
$$= 3x^3y^3 \neq 0$$

Therefore, the integrating factor u is

$$u = \frac{1}{3x^3y^3}, \qquad \because u = \frac{1}{xM - yN}$$

Now multiplying the given equation by the integrating factor, we obtain

$$\frac{1}{3}\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \frac{1}{3}\left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$$

Therefore, solutions of the given differential equation are given by

$$-\frac{1}{xy} + 2\ln|x| - \ln|y| = C$$

where $3C_0 = C$