MTH 401 – DIFFERENTIAL EQUATIONS

LECTURE 1: INTRODUCTION TO DIFFERENTIAL EQUATIONS

Background

Linear $y=mx+c$

Quadratic $ax2+bx+c=0$

Cubic $ax3+bx2+cx+d=0$

Systems of Linear equations

$ax+by+c=0$

$lx+my+n=0$

Solution ?

Equation

Differential Operator

$$\frac{dy}{dx}=\frac{1}{x}$$

Taking anti derivative on both sides

$y=\ln x$

From the past

- *Algebra*
- *Trigonometry*
- *Calculus*
- *Differentiation*
- *Integration*

- *Differentiation*
  - *Algebraic Functions*
  - *Trigonometric Functions*
  - *Logarithmic Functions*
  - *Exponential Functions*
  - *Inverse Trigonometric Functions*

- More Differentiation
  - Successive Differentiation
  - Higher Order
  - Leibnitz Theorem
- Applications
  - Maxima and Minima

- Tangent and Normal

- Partial Derivatives

y=f(x)

f(x,y)=0

z=f(x,y)

Integration

- Reverse of Differentiation
- By parts
- By substitution
- By Partial Fractions
- Reduction Formula

Frequently required

- Standard Differentiation formulae
- Standard Integration Formulae

Differential Equations

- Something New
- Mostly old stuff
    - Presented differently
    - Analyzed differently
    - Applied Differently

$$\frac{dy}{dx} - 5y = 1$$

$$(y-x)\,dx + 4x\,dy = 0$$

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial y} = u$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} = 0$$

Fundamentals

- Definition of a differential equation.
- Classification of differential equations.
- Solution of a differential equation.
- Initial value problems associated to DE.
- Existence and uniqueness of solutions

Elements of the Theory

- Applicable to:
  - Chemistry
  - Physics
  - Engineering
  - Medicine
  - Biology
  - Anthropology
- Differential Equation – involves an unknown function with one or more of its derivatives
- Ordinary D.E. – a function where the unknown is dependent upon only one independent variable

Examples of DEs

$$\frac{dy}{dx}-5y = 1$$

$$(y-x)dx+4xdy = 0$$

$$\frac{d^2y}{dx^2}+5\left(\frac{dy}{dx}\right)^3-4y = e^x$$

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} = 0$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial v}{\partial y} = u$$

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}+2\frac{\partial u}{\partial t} = 0$$

Specific Examples of ODE's

$$\frac{du}{dt} = F(t)G(u), \qquad \text{the growth equation;}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin(\theta) = F(t), \qquad \text{the pendulum equation;}$$

$$\frac{d^2y}{dt^2} + \varepsilon(y^2+1)\frac{dy}{dt} + y = 0, \qquad \text{the van der Pol equation;}$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t), \qquad \text{the LCR oscillator equation;}$$

$$\frac{dp}{dt} = -2a(t)p + \frac{b(t)^2}{u(t)}p^2 - v(t), \qquad \text{a Riccati equation.}$$

- The order of an equation:
  - The order of the highest derivative appearing in the equation

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y \qquad = e^x$$

$$a^2\frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} \qquad = 0$$

Ordinary Differential Equation

If an equation contains only ordinary derivatives of one or more dependent variables, *w.r.t* a single variable, then it is said to be an Ordinary Differential Equation **(ODE)**. For example the differential equation

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

is an ordinary differential equation.

Partial Differential Equation

Similarly an equation that involves partial derivatives of one or more dependent variables *w.r.t* two or more independent variables is called a Partial Differential Equation **(PDE)**. For example the equation

$$a^2\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u}{\partial x^2} = 0$$

is a partial differential equation.

Results from ODE data

- The solution of a general differential equation:
  - $f(t, y, y', \ldots, y(n)) = 0$
  - is defined over some interval I having the following properties:
    - $y(t)$ and its first n derivatives exist for all t in I so that $y(t)$ and its first n - 1 derivates must be continuous in I
    - $y(t)$ satisfies the differential equation for all t in I

- General Solution – all solutions to the differential equation can be represented in this form for all constants
- Particular Solution – contains no arbitrary constants
- Initial Condition
- Boundary Condition
- Initial Value Problem (IVP)
- Boundary Value Problem (BVP)

IVP Examples

- The Logistic Equation
  - $p' = ap - bp2$
  - with initial condition $p(t0) = p0$; for $p0 = 10$ the solution is:
  - $p(t) = 10a / (10b + (a - 10b)e\text{-}a(t\text{-}t0))$
- The mass-spring system equation
  - $x'' + (a / m)\, x' + (k / m)x = g + (\mathrm{F}(t) / m)$

BVP Examples

- Differential equations
  - $y'' + 9y = \sin(t)$
    - with initial conditions $y(0) = 1, y'(2\mathrm{p}) = -1$
    - $y(t) = (1/8)\sin(t) + \cos(3t) + \sin(3t)$
  - $y'' + \mathrm{p}2y = 0$
    - with initial conditions $y(0) = 2, y(1) = -2$
    - $y(t) = 2\cos(\mathrm{p}t) + (c)\sin(\mathrm{p}t)$

Properties of ODE's

- Linear – if the nth-order differential equation can be written:
    - $an(t)y(n) + an-1(t)y(n-1) + \ldots + a1y' + a0(t)y = h(t)$

- Nonlinear – not linear

  $x3(y''')3-x2y(y'')2+3xy'+5y=ex$

Superposition

- Superposition – allows us to decompose a problem into smaller, simpler parts and then combine them to find a solution to the original problem.

Explicit Solution

A solution of a differential equation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^2y}{dx^2}\right) = 0$$

that can be written as $y = f(x)$ is known as an explicit solution .

Example: The solution $\boldsymbol{y = xex}$ is an explicit solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

Implicit Solution

A relation $\boldsymbol{G(x,y)}$ is known as an implicit solution of a differential equation, if it defines one or more explicit solution on I.

Example: The solution $x2 + y2 - 4=0$ is an implicit solution of the equation $y' = -x/y$ as it defines two explicit solutions $y=\underline{+}(4-x2)1/2$