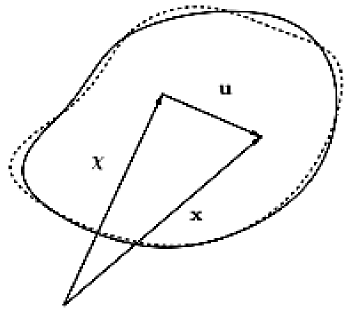
LECTURE 8: LINEAR ELASTICITY

■ **Overview** We show how to obtain the equations of linear elasticity by linearising the general nonlinear equations of elasticity for small displacements. We also consider the solution of simple problems.

6.1 Infinitesimal strain tensor

The central object is not the mapping χ but the displacement.



$$\mathbf{u} = \mathbf{x} - \mathbf{X} = \chi(\mathbf{X}, t) - \mathbf{X}$$
$$\Longrightarrow \nabla\mathbf{u} = \text{Grad}\,\chi - \mathbf{1} = \mathbf{H} = \mathbf{F} - \mathbf{1},$$

the displacement gradient.

Assumptions:

- Displacement gradient is small.

Now consider the strain tensor,

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{1}\right)$$

$$\mathbf{F} = \mathbf{1} + \mathbf{H} \implies \mathbf{E} = \frac{1}{2}\left((\mathbf{1}+\mathbf{H})\left(\mathbf{1}+\mathbf{H}^T\right) - \mathbf{1}\right) = \underbrace{\frac{1}{2}(\mathbf{H}+\mathbf{H}^T)}_{\mathbf{E}} + \mathcal{O}(\mathbf{H}^2)$$

$$\mathbf{E} = \frac{1}{2}(\mathbf{H}+\mathbf{H}^T) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$$

6.2 Constant relationships

$$\mathbf{S} = \mathcal{S}(\mathbf{F}), \qquad \mathbf{T} = \mathcal{T}(\mathbf{F}).$$

We assume $\mathcal{S}(\mathbf{1}) = 0$ (no residual stress).

$$\Longrightarrow \mathbf{S} = \mathcal{S}(\mathbf{1}+\mathbf{H}) = \underbrace{\mathcal{S}(\mathbf{1})}_{0} + \underbrace{D\mathcal{S}(\mathbf{1})[\mathbf{H}]}_{\mathcal{C}[\mathbf{H}]} + \mathcal{O}(\mathbf{H}^2),$$

where C is linear in $\mathbf{H}$.

$$\mathbf{T} = \underbrace{\mathcal{T}(1)}_{\mathbf{0}} + D\mathcal{T}(1)[\mathbf{H}] + \mathcal{O}(\mathbf{H}^2)$$

Which one to use?

$$\mathbf{T} = J^{-1}\mathbf{FS}$$

$$\implies \mathcal{T} = J^{-1}\mathbf{F}\mathcal{S}$$

and

$$\mathcal{T} = D\mathcal{T}[\mathbf{H}] = J^{-1}(1+\mathbf{H})D\mathcal{S}(1)[\mathbf{H}] = \det(1+\mathbf{H})(1+\mathbf{H})D\mathcal{S}(1)[\mathbf{H}] + \mathcal{O}(\mathbf{H}^2)$$

$$\implies D\mathcal{T}[\mathbf{H}] = D\mathcal{S}[\mathbf{H}] = D\mathcal{S}[\mathbf{H}] = C[\mathbf{H}]$$

C elasticity tensor,

$$T_{ij} = C_{ijk\ell}H_{k\ell}$$

Major symmetries:

$$C_{ijk\ell} = C_{k\ell ij}$$

Minor symmetries:

$$C_{ijk\ell} = C_{ij\ell k} = C_{jik\ell} = C_{ji\ell k}$$

$\implies$ from 81 components to 36 independent components.

Note also

$$T_{ij} = C_{ijk\ell}\Big[\underbrace{\frac{1}{2}(\mathbf{H}+\mathbf{H}^T)}_{\mathbf{E}} + \frac{1}{2}(\mathbf{H}-\mathbf{H}^T)\Big]_{k\ell}$$

$$\boxed{T_{ij} = C_{ijk\ell}\mathrm{E}_{k\ell}}$$

constant relationship for linear elasticity.

6.3 Isotropic linear elasticity

If the material is isotropic:

$$S_{ij} = T_{ij} = 2\mu e_{ij} + \lambda(\mathrm{tr}\ \mathbf{E})\delta_{ij}$$

where

$$C_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + 2\mu\delta_{ik}\delta_{j\ell}$$

for μ and λ the Lamé coefficients. From the symmetry of C and positive definiteness, we have

$$\mu > 0, \qquad 2\mu + 3\lambda > 0.$$

Note: C is positive definite means

$$\mathbf{E}.\mathsf{C}(\mathbf{E}) > 0, \qquad \forall\mathbf{E} \in \mathrm{Sym}.$$

Material hyperelasticity $\iff$ C is positive definite
$\implies$ C is symmetric.

If the body is *homogeneous*, then ρ_0, λ, μ are constant.

6.3.1 Equations:

$\mathbf{u} = (\chi)(\mathbf{X}) - \mathbf{X}$.

$$\mathbf{S} = \mathbb{C}[\mathbf{E}], \qquad e = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$$

$$\text{Div } \mathbf{S} + \mathbf{b}_0 = \rho_0\ddot{\mathbf{u}} = \rho_0\mathbf{a}$$

Assume homogeneity and isotropy,

$$\mathbf{S} = 2\mu\mathbf{E} + \lambda(\text{tr } \mathbf{E})\mathbf{1}.$$

$$\begin{aligned}\text{Div } \mathbf{S} &= \not{2}\mu\text{Div}\left(\frac{1}{\not{2}}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)\right) + \lambda\text{Div}\left(\text{tr } \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)\mathbf{1}\right)\\ &= \mu\Delta\mathbf{u} + \mu\text{Grad Div } \mathbf{u} + \lambda\text{Grad Div } \mathbf{u}\\ &= \mu\Delta\mathbf{u} + (\mu + \lambda)\text{Grad Div } \mathbf{u}\end{aligned}$$

Therefore we have the *Navier equation*,

$$\boxed{\mu\Delta\mathbf{u} + (\mu + \lambda)\text{Grad Div } \mathbf{u} + \mathbf{b}_0 = \rho_0\ddot{\mathbf{u}}}$$

Note that $\mathbf{u} = \mathbf{u}(\mathbf{X})$ implies that $\mathbf{x}$ does not appear any more (we can replace $\mathbf{X}$ by $\mathbf{x}$ if we want – I don't).

In components,

$$\mathbf{u} = \mathbf{u}(\mathbf{X}, t) = u_i\mathbf{E}_i$$

implies

$$\boxed{\rho_0\frac{\partial^2 u_i}{\partial t^2} = b_i + \mu\sum_{j=1}^{3}\frac{\partial^2 u_i}{\partial X_j^2} + (\lambda + \mu)\frac{\partial^2 u_j}{\partial X_i \partial X_j}}$$

6.4 Examples

To understand the meaning of the elastic moduli, we consider simple deformations.

1) Pure shear, $\mathbf{u} = (\gamma X_2, 0, 0)$ [pic]

$$[\mathbf{E}] = \frac{1}{2}\begin{bmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [\sigma] = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

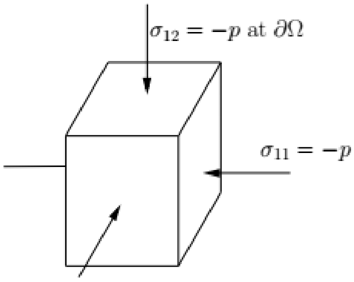
$\implies \tau = \mu\gamma \implies \mu$ is the shear modulus.
(Care! Normal stresses vanish... and remember the condition $\mu > 0$!)

2) Uniform compression, $\mathbf{u} = \delta\mathbf{X}$ and $\mathbf{u} = \mathbf{x} - \mathbf{X} = (\delta + 1)\mathbf{X} - \mathbf{X}$

$$\mathbf{E} = \delta\mathbf{1}, \qquad \sigma = -p\mathbf{1}$$

We use

$$\mathbf{E} = \frac{1}{2\mu}\left[\sigma - \frac{\lambda}{2\mu + 3\lambda}(\text{tr } \sigma)\mathbf{1}\right]$$

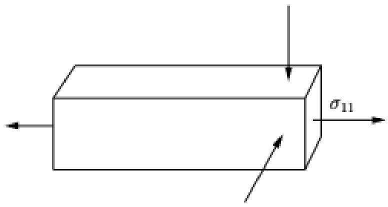


$$\begin{aligned}\delta 1 &= \frac{1}{2\mu}\left[-p1 + \frac{\lambda}{2\mu+3\lambda}3p1\right]\\ &= \frac{1}{2\mu}p\left[\frac{-(2\mu+3\lambda)+3\lambda}{2\mu+3\lambda}\right]1\\ &= -\frac{p}{2\mu+3\lambda}\end{aligned}$$

$$\implies p = -(2\mu+3\lambda)\delta = -3\underbrace{\left(\frac{2\mu+3\lambda}{3}\right)}_{\kappa}\delta,$$

where κ is the *Modulus of Compression*. Remember the condition $2\mu + 3\lambda > 0$!

3) Uniaxial tension, $\sigma = t\mathbf{E}_1 \otimes \mathbf{E}_1$



$$[\mathbf{E}] = \text{diag}(\alpha, \beta, \beta), \qquad \alpha = \frac{t}{E}, \qquad \beta = -\nu\alpha.$$

$$E = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda}, \qquad \nu = \frac{\lambda}{2(\mu+\lambda)}$$

Here E is equated to the *infinitesimal Young's modulus* and ν is equated to *Poisson's ratio.*

$$\boxed{\mathbf{E} = \frac{1}{E}((1+\nu)\sigma - \nu(\text{tr } \sigma)1)}$$

an alternative form for $\mathbf{E}$.

Expect $\nu > 0$ (Care! It's an auxetic material!)

Now

$$\kappa = \frac{2\mu+3\lambda}{3} = \frac{E}{3(1-2\nu)},$$

so that as $\nu \to 1/2$, $\kappa \to \infty$, and we would need an infinite force to change the volume. Therefore incompressible materials have $\nu = 1/2$.