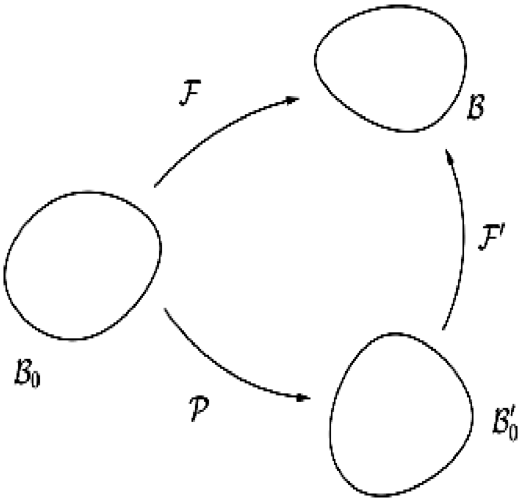
ENG 302 –SOLID MECHANICS

LECTURE 7: CONSTITUTIVE EQUATIONS-CONTINUED



4.4 Hyperelastic isotropic materials

Isotropy is a particularly important material symmetry. If $\mathcal{T}(\mathbf{FQ}) = \mathcal{T}(\mathbf{F})$, $\forall\mathbf{Q}$, proper orthonormal, then the material is isotropic and $SO(3)$ is its material symmetry group.

Response of the material independent of its orientation (this is a pointwise notion!) $\Longleftrightarrow$ there is no physical way to distinguish the orientation of a local element. For a hyperelastic material (isotropic), we have

$$\mathsf{W}(\mathbf{FP}) = \mathsf{W}(\mathbf{F}), \quad \forall\mathbf{P}, \text{ rotations.}$$

So we have

$$\mathsf{W}(\mathbf{QF}) = \mathsf{W}(\mathbf{F}), \qquad \mathsf{W}(\mathbf{FP}) = \mathsf{W}(\mathbf{F}).$$

Use $\mathbf{P} = \mathbf{RQ}^T$, then

$$\mathsf{W}(\mathbf{QF}) = \mathsf{W}(\mathbf{PR}^T\mathbf{RUP}) = \mathsf{W}(\mathbf{F}) = \mathsf{W}(\mathbf{U}) = \mathsf{W}(\mathbf{P}^T\mathbf{UP}), \quad \forall\mathbf{P},$$

and

$$\mathsf{W}(\mathbf{QF}) = \mathsf{W}(\mathbf{QFRQ}^T) = \mathsf{W}(\mathbf{QVQ}^T) = \mathsf{W}(\mathbf{V})$$

$$\Longrightarrow \boxed{\mathsf{W}(\mathbf{QFQ}^T) = \mathsf{W}(\mathbf{V}), \qquad \mathsf{W}(\mathbf{PVP}^T) = \mathsf{W}(\mathbf{U})}$$

$\Longleftrightarrow$ W is an isotropic function of $\mathbf{V}$ or $\mathbf{U}$.

See linear algebra: $\det(\mathbf{QMQ}^T) = \det(\mathbf{M})$ and $\operatorname{tr}(\mathbf{QMQ}^T) = \operatorname{tr}(\mathbf{Q})$.

There exist only three functionally independent invariant functions of a 3×3 second-order tensor. Let λ_1, λ_2, λ_3, be the eigenvalues of $\mathbf{V}$, *i.e.*the principal stretches.

$$I_1 = \operatorname{tr}(\mathbf{V}^2) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad \lambda_i > 0$$
$$I_2 = \frac{1}{2}\left[I_1^2 - \operatorname{tr}(\mathbf{V}^4)\right] = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$$
$$I_3 = \det(\mathbf{V}^2) = \lambda_1^2\lambda_2^2\lambda_3^2 = J^2$$

I_1, I_2, I_3 are symmetric functions of stretches. Of course $\lambda^6 - I_1\lambda^4 + I_2\lambda^2 - I_3 = 0$, Cayley's theorem.

$$\Longrightarrow \mathsf{W}(\mathbf{F}) = \overline{\mathsf{W}}(I_1, I_2, I_3) = \widehat{\mathsf{W}}(\lambda_1, \lambda_2, \lambda_3).$$

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Contradiction, $\mathsf{W}(\mathbf{1}) = 0 \implies \overline{\mathsf{W}}(3,3,1) = 0,\ \widehat{\mathsf{W}}(1,1,1) = 0.$

Let us now express the stress tensors in terms of W and its derivatives.

$$\mathbf{S} = \frac{\partial \mathsf{W}}{\partial \mathbf{F}}$$

Let $\mathsf{W} = \mathsf{W}(\mathbf{V}^2) = \mathsf{W}(\mathbf{B})$,

$$\implies \mathbf{S} = \frac{\partial \mathsf{W}(\mathbf{B})}{\partial \mathbf{F}} = 2\mathbf{F}\frac{\partial \mathsf{W}}{\partial \mathbf{B}} = J\mathbf{F}^{-1}\mathbf{T}$$

$$\mathbf{T} = 2J^{-1}\mathbf{F}^{-1}\frac{\partial \mathsf{W}}{\partial \mathbf{F}}\mathbf{F}$$

$$\mathbf{T} = J^{-1}\mathbf{F}\mathbf{S}$$

$$\mathbf{S} = 2\frac{\partial \mathsf{W}}{\partial \mathbf{C}}\mathbf{F}^T, \qquad \mathbf{T} = 2J^{-1}\mathbf{F}^{-1}\frac{\partial \mathsf{W}}{\partial \mathbf{C}}\mathbf{F}^T$$

$$S = 2J^{-1}\mathbf{F}\mathbf{F}^T\frac{\partial \mathsf{W}}{\partial \mathbf{B}} = 2J^{-1}\mathbf{B}\frac{\partial \mathsf{W}}{\partial \mathbf{B}}$$

Now for $\partial \mathsf{W}/\partial \mathbf{B}$,

$$\mathsf{W} = \overline{\mathsf{W}}(I_1, I_2, I_3)$$

where $I_1 = \mathrm{tr}(\mathbf{B})$, $I_2 = \frac{1}{2}(I_1^2 - \mathrm{tr}(\mathbf{B}^2))$ and $I_3 = \mathrm{Det}(\mathbf{B})$.

Some interesting results

$$\frac{\partial \mathrm{tr}\mathbf{B}}{\partial \mathbf{B}} = \mathbf{1} = \frac{\partial I_1}{\partial \mathbf{B}}$$

$$\frac{\partial \det \mathbf{B}}{\partial \mathbf{B}} = \det \mathbf{B}\ \mathbf{B}^{-1} = \frac{\partial I_3}{\partial \mathbf{B}}$$

$$\frac{\partial I_2}{\partial \mathbf{C}} = \frac{1}{2}.2\,(\mathrm{tr}\mathbf{B})\,\mathbf{1} - \frac{\partial \mathrm{tr}\mathbf{B}^2}{\partial \mathbf{B}} = (\mathrm{tr}\mathbf{B})\,\mathbf{1} - 2\mathbf{B}\frac{\partial \mathrm{tr}\mathbf{B}}{\partial \mathbf{B}} = I_1\mathbf{1} - \mathbf{B}$$

$$\implies \mathbf{S} = 2J^{-1}\mathbf{B}\left[\frac{\partial \mathsf{W}}{\partial I_1}\mathbf{1} + \frac{\partial \mathsf{W}}{\partial I_2}(I_2\mathbf{1} - \mathbf{B}) + \frac{\partial \mathsf{W}}{\partial I_3}J\mathbf{B}^{-1}\right] = \mathcal{W}_0\mathbf{1} + \mathcal{W}_1\mathbf{B} + \mathcal{W}_2\mathbf{B}^2,$$

where

$$\mathcal{W}_0 = 2\frac{\partial \mathsf{W}}{\partial I_3}, \qquad \mathcal{W}_1 = 2J^{-1}\frac{\partial \mathsf{W}}{\partial I_1} + 2J^{-1}\frac{\partial \mathsf{W}}{\partial I_2}I_1, \qquad \mathcal{W}_2 = -2J^{-1}\frac{\partial \mathsf{W}}{\partial I_2}$$

and $\mathcal{W}_i = \mathcal{W}_i(I_1, I_2, I_3)$.

Note that we can also write

$$\mathbf{T} = \sum t_i \mathbf{v}_i \otimes \mathbf{v}_i,$$

since $\mathbf{T}$ is coaxial with $\mathbf{V}$. ($\mathbf{v}_i$ principal direction.) Then

$$t_i = J^{-1}\lambda_i\frac{\partial \widehat{\mathsf{W}}}{\partial \lambda_i}, \qquad \widehat{\mathsf{W}} = \mathsf{W}(\lambda_1, \lambda_2, \lambda_3), \qquad J = \lambda_1\lambda_2\lambda_3.$$

Similarly

$$\mathbf{S} = \sum s_i \mathbf{u}^{(1)} \otimes \mathbf{v}^{(1)}, \qquad s_i = \frac{\partial \widehat{\mathsf{W}}}{\partial \lambda_i}$$

4.5 Hyperelastic incompressible materials

In general assume that we want to restrict the set of possible deformations.

$$\mathcal{C}(\mathbf{F}) = 0$$

e.g. $\det \mathbf{F} - 1 = 0$, incompressibility.

We introduce a Lagrangian multiplier, p,

$$\mathsf{W} \to \mathsf{W} - p\mathcal{C}$$

Again by variational calculus,

$$\operatorname{tr}\left[\left(\frac{\partial}{\partial \mathbf{F}}(\mathsf{W} - p\mathcal{C}) - \mathbf{S}\right)\dot{\mathbf{F}}\right] = 0$$

$$\implies \mathbf{S} = \frac{\partial \mathsf{W}}{\partial \mathbf{F}} - p\frac{\partial \mathcal{C}}{\partial \mathbf{F}}$$

e.g. $\mathcal{C} = \det \mathbf{F} - 1$, so that $\frac{\partial \mathcal{C}}{\partial \mathbf{F}} = \det \mathbf{F}\ \mathbf{F}^{-1} = J\mathbf{F}^{-1}$ and

$$\mathbf{S} = \frac{\partial \mathsf{W}}{\partial \mathbf{F}} - J\mathbf{F}^{-1} \quad \implies \quad \mathbf{T} = J^{-1}\mathbf{F}\mathbf{S} = J^{-1}\mathbf{F}\frac{\partial \mathsf{W}}{\partial \mathbf{F}} - p\mathbf{1}$$

4.6 Choice of strain-energy density for hyperelastic materials

Incompressible if $f(J) = 0$

neo-Hookean	$\mathsf{W} = \frac{\mu}{2}(I_1 - 3) + f(J)$
Mooney–Rivlin	$\mathsf{W} = c_1(I_1 - 3) + c_2(I_2 - 3) + f(J)$
Varga	$\mathsf{W} = 2\mu(\lambda_1 + \lambda_2 + \lambda_3 - 3) + f(J)$
Ogden	$\mathsf{W} = \sum_{p=1}^{\infty} \mu_p(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p}) + f(J)$
Fung	$\mathsf{W} = \frac{1}{2\alpha}\left[e^{\alpha(I_1 - 3)} - 1\right] + f(J)$ where $3 < \alpha < 20$
Gent	$\mathsf{W} = -\frac{1}{2\beta}\ln\left[1 - \beta(I_1 - 3)\right] + f(J)$ where $0.4 < \beta < 3$

Possible choices for $f(J)$: $\mu_1(I_3 - 1)$, $\mu_2(J - 1)^2$, $\mu_2 \ln I_3$, $\mu_2 \ln J$, $\ldots$