LECTURE 3: THE DEFORMATION GRADIENT F

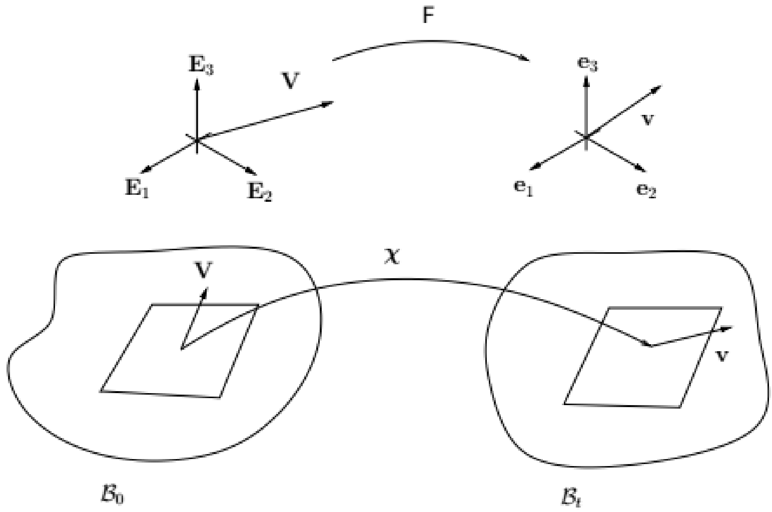
To have a notion of local deformation, we define,

$$\mathbf{F}(\mathbf{x},t) = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \text{Grad } \mathbf{x} = \text{Grad } \boldsymbol{\chi}(\mathbf{X},t).$$

Notice the upper case on Grad to denote the fact that we take derivatives w.r.t. $\mathbf{X}$. Explicitly, we have

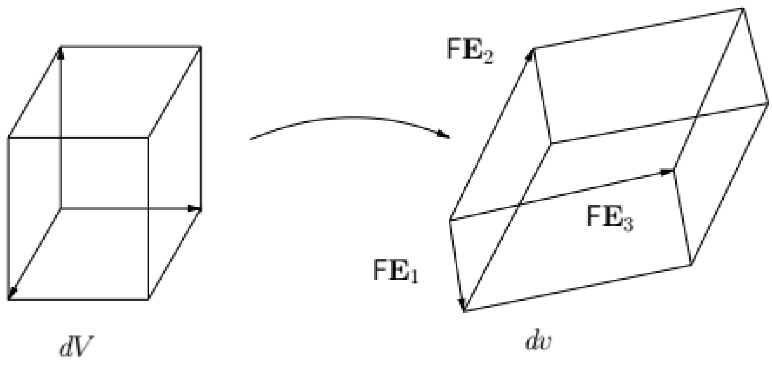
$$\mathbf{F} = \frac{\partial}{\partial x_j}(x_i \mathbf{e}_i) \otimes \mathbf{E}_j = \frac{\partial x_i}{\partial X_j} \mathbf{e}_i \otimes \mathbf{E}_j \equiv F_{ij} \mathbf{e}_i \otimes \mathbf{E}_j,$$

so the last tensor product is indeed mixed. Geometrically, $\mathbf{F}$ is a linear map that transforms



a vector $\mathbf{v}$ defined in the tangent space $T_p\mathcal{B}_0$ at a point p in the reference configuration to a vector in the tangent space $T_p\mathcal{B}$ at the same material point p but in the current configuration.

2.4.1 Transformation of volume



Now consider the image of an infinitesimal volume element of size $\mathrm{d}\lambda$ along three unit vectors $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$. Each unit vector transforms as

$$\mathbf{F}\mathbf{E}_k = F_{ij}\left(\mathbf{e}_i \otimes \mathbf{E}_j\right)\mathbf{E}_k = F_{ij}(\mathbf{E}_k \cdot \mathbf{E}_j)\mathbf{e}_i = F_{ij}\delta_{kj}\mathbf{e}_i = F_{ik}\mathbf{e}_i.$$

Note that the column k of F is the image of the k$^{\text{th}}$ basis vector as found in regular matrix algebra.

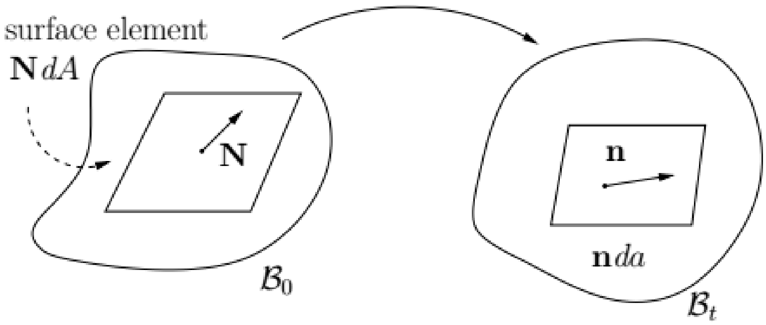
The infinitesimal volume is

$$\mathrm{d}v = (d\lambda)^3 \det(\mathbf{F}\mathbf{E}_1, \mathbf{F}\mathbf{E}_2, \mathbf{F}\mathbf{E}_3) = (d\lambda)^3 \det \mathbf{F} \det(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3) = \det \mathbf{F} \mathrm{d}V.$$

We define $J = \det \mathbf{F}$ and we see that it has a natural description as the change of volume at a material point p.

2.4.2 Transformation of area: Nanson's formula.

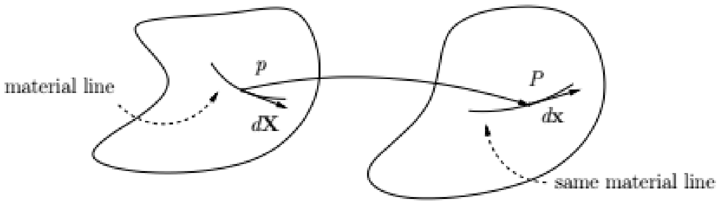
The following rule is for the corresponding change in a local area element defined at a point p through the normal vector $\mathbf{n}$ (see Problem Sheets).



$$\mathbf{n}\mathrm{d}a = J\mathbf{F}^{-T}\mathbf{N}\mathrm{d}A \quad \Longleftrightarrow \quad \mathrm{d}\mathbf{a} = J\mathbf{F}^{-T}\mathrm{d}\mathbf{A}$$

2.4.3 Transformation of line element.

We have assumed $\mathbf{F}$ is a bijection $\implies \det \mathbf{F} > 0$, so that $\mathbf{F}^{-1}$ is well defined. Now take a local infinitesimal line element in $\mathcal{B}_0$ at p.



$$\implies \quad \mathrm{d}\mathbf{x} = \mathbf{F}\mathrm{d}\mathbf{X}.$$

Let $\mathbf{M}$ be a unit vector along $\mathrm{d}\mathbf{X}$:

$$\mathrm{d}\mathbf{X} = \mathbf{M}\mathrm{d}S = \mathbf{M}|\mathrm{d}\mathbf{X}|$$
$$\mathrm{d}\mathbf{x} = \mathbf{m}\mathrm{d}s = \mathbf{m}|\mathrm{d}\mathbf{x}|$$

$\implies \mathbf{m}\mathrm{d}s = \mathbf{F}\mathbf{M}\mathrm{d}s.$

Take the norm of each side:

$$|\mathrm{d}s|^2 = (\mathbf{F}\mathbf{M} \cdot \mathbf{F}\mathbf{M})|\mathrm{d}S|^2 = (\mathbf{F}^T\mathbf{F}\mathbf{M}) \cdot \mathbf{M}|\mathrm{d}S|^2$$
$$\Longleftrightarrow \frac{\mathrm{d}s}{\mathrm{d}S} = \sqrt{(\mathbf{F}^T\mathbf{F}\mathbf{M}) \cdot M} \equiv \lambda(M),$$

where $\mathrm{d}s/\mathrm{d}S$ is the change of length of a material line in the direction M and λ is a stretch. This implies that

$$\frac{\mathrm{d}s}{\mathrm{d}S} = 1 \quad \Longleftrightarrow \quad s = S, \ \forall M \quad \Longleftrightarrow \quad \mathbf{F}^T\mathbf{F} = \mathbf{1}.$$

A material is *unstrained* if $\mathbf{F}^T\mathbf{F} = \mathbf{1}$. However, the converse is not true. Therefore, we cannot use $\mathbf{F}$ directly to measure the strain in a body.

2.4.4 Polar decomposition theorem.

The action of $\mathbf{F}$ is to rotate a vector along a direction M to a direction N, then stretch it to a size $\lambda(M)$.

$$\Longrightarrow \text{stretch} + \text{rotation}.$$

This is contained in the polar decomposition theorem:

Theorem Let $\mathbf{F}$ be a second-order Cartesian tensor such that $\det \mathbf{F} > 0$. Then $\exists$ unique positive definite symmetric tensors $\mathbf{U}$, $\mathbf{V}$ and a unique proper orthogonal tensor $\mathbf{R}$ such that,

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}.$$

Note

$$\begin{aligned} \mathbf{F}^T\mathbf{F} &= \mathbf{U}^2 \equiv \mathbf{C}, \qquad \text{right Cauchy Green tensor} \\ \mathbf{F}\mathbf{F}^T &= \mathbf{V}^2 \equiv \mathbf{B}, \qquad \text{left Cauchy Green tensor} \end{aligned}$$

Explicitly, we have

$$\begin{aligned} \mathbf{F} &= F_{ij}\mathbf{e}_i \otimes \mathbf{E}_j \\ \mathbf{F}^T &= F_{k\ell}\mathbf{E}_k \otimes \mathbf{e}_\ell \\ \mathbf{F}^T\mathbf{F} &= (F_{ki}\mathbf{E}_k \otimes \mathbf{e}_i)(F_{ij}\mathbf{e}_i \otimes \mathbf{E}_j) \\ &= \left(\mathbf{F}^T\mathbf{F}\right)_{kj} \mathbf{E}_k \otimes \mathbf{E}_j = \mathbf{U}^2_{kj}\mathbf{E}_k \otimes \mathbf{E}_j, \quad \text{Lagrangian} \\ \mathbf{F}\mathbf{F}^T &= \left(\mathbf{F}\mathbf{F}^T\right)_{ij} \mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{V}^2_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \quad \text{Eulerian} \end{aligned}$$

2.4.5 Strain tensors.

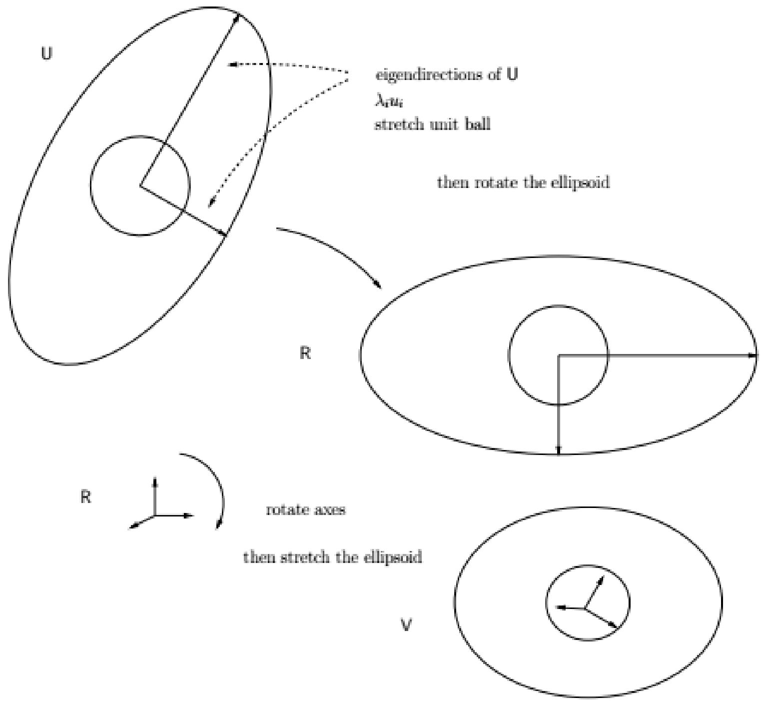
We need to define a notion of strain. A strain tensor is a tensor that is identically zero if there is no stretch in any direction of the body. For instance,

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{1}\right), \qquad \text{Green strain tensor}$$

It is easy to check that for a global rotation, translation or rigid body motion we have

$$\mathbf{E} = 0.$$

Only for stretching is $\mathbf{E} \neq \mathbf{0}$ and depending on the direction of space, different stretches are obtained.



Other possible strain tensors can be constructed

$$\mathbf{E}^{(\alpha)} = \frac{1}{2}\left(\mathbf{U}^{\alpha} - \mathbb{1}\right), \qquad \alpha \neq 0, \quad \alpha \in \mathbb{R}^{+}$$
$$\mathbf{E}^{(0)} = \ln \mathbf{U}.$$

or

$$\mathbf{e}^{(\alpha)} = \frac{1}{2}\left(\mathbf{V}^{\alpha} - \mathbb{1}\right), \qquad \alpha \neq 0, \quad \alpha \in \mathbb{R}^{+}$$
$$\mathbf{e}^{(0)} = \ln \mathbf{V}.$$

These are, respectively, Lagrangian (upper case, they live in the reference configuration) and Eulerian strain tensors (lower case, they live in the current configurations).

2.4.6 The displacement gradient.

Displacement is an important notion in linear elasticity. It is defined as

$$\mathbf{u} = \mathbf{x} - \mathbf{X} \quad \Longrightarrow \quad \mathbf{x} = \mathbf{X} + \mathbf{u}$$
$$\Longrightarrow \ \mathbf{F} = \text{Grad } \mathbf{x} = \mathbb{1} + \text{Grad } \mathbf{u},$$

where Grad $\mathbf{u}$ is the displacement gradient.

2.4.7 The velocity gradient tensor.

Another important kinematic quantity is *the velocity gradient tensor*, defined as

$$\mathbf{L} = \text{grad } \mathbf{v}, \qquad L_{ij} = \frac{\partial v_i}{\partial x_j}, \qquad \mathbf{L} = L_{ij}\mathbf{e}_i \otimes \mathbf{e}_j.$$

Since Grad $\mathbf{u} = (\text{grad } \mathbf{u})\mathbf{F}$ by the chain rule,

$$\text{Grad } \mathbf{v} = (\text{grad } \mathbf{v})\mathbf{F} = \mathbf{LF},$$

but also,

$$\text{Grad } \mathbf{v} = \text{Grad } \dot{\mathbf{x}} = \frac{\partial}{\partial t}\text{Grad } \mathbf{x} = \frac{\partial \mathbf{F}}{\partial t} = \dot{\mathbf{F}},$$

so that

$$\boxed{\dot{\mathbf{F}} = \mathbf{LF}}$$

and (see Problem Sheets)

$$\boxed{\dot{\mathbf{J}} = \mathbf{J}\,\text{div } \mathbf{v}}$$

Another way to obtain this result is to use the formula for the derivative of a determinant.

$$\frac{\partial}{\partial t}\det \mathbf{F} = (\det \mathbf{F})\,\text{tr}(\mathbf{F}^{-1}\dot{\mathbf{F}}) = (\det \mathbf{F})\,\text{tr}(\mathbf{L}),$$

So that,

$$\Longrightarrow \quad \dot{\mathbf{J}} = \mathbf{J}\,\text{tr}(\mathbf{L}) = \mathbf{J}\,\text{div } \mathbf{v}.$$

We define an *isochoric deformation* to be one such that there is no change of local volume element, that is, $\mathbf{J} \equiv 1$, $\dot{\mathbf{J}} = 0 \implies \boxed{\text{div } \mathbf{v} = 0}$, (this is the incompressibility conditions found in fluids).

2.5 Examples of deformation

2.5.1 Homogeneous deformation

$$\mathbf{x} = \mathbf{FX} + \mathbf{x}, \qquad \mathbf{F} \text{ constant}$$

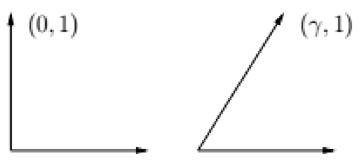
- *Simple elongation*

$$\mathbf{F} = \mathbf{U} = \lambda_1 \mathbf{U}^{(1)} \otimes \mathbf{U}^{(1)} + \lambda_2 \left(\mathbf{U}^{(2)} \otimes \mathbf{U}^{(2)} + \mathbf{U}^{(3)} \otimes \mathbf{U}^{(3)}\right)$$

- *Dilation*

$$\mathbf{F} = \lambda\,\mathbb{1}$$

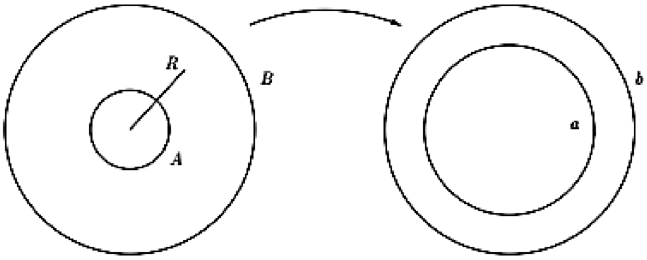
- *Simple shear*



$$x_1 = X_1 + \gamma X_2, \qquad x_2 = X_2, \qquad x_3 = X_3,$$

which imply (homework),

$$\Longrightarrow \quad \mathbf{F} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{U}^2 = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



2.5.2 Inflation of a spherical shell

Symmetric inflation

Define a function $r = f(R)$ such that $r(A) = a$, $r(B) = b$, and

$$r = f(R), \qquad \theta = \Theta, \qquad \phi = \Phi.$$

Then

$$\mathbf{X} = R\mathbf{e}_R, \qquad \mathbf{x} = r\mathbf{e}_r = f(R)\mathbf{e}_r,$$

and

$$\begin{aligned}\frac{\partial \mathbf{x}}{\partial \mathbf{X}} &= \frac{\partial \mathbf{x}}{\partial R} \otimes \mathbf{e}_R + \frac{\partial \mathbf{x}}{\partial \Theta} \otimes \mathbf{e}_\Theta + \frac{\partial \mathbf{x}}{\partial \Phi} \otimes \mathbf{e}_\Phi \\ &= \frac{\partial (r\mathbf{e}_r)}{\partial R} \otimes \mathbf{e}_R + \frac{\partial (r\mathbf{e}_r)}{\partial \Theta} \otimes \mathbf{e}_\Theta + \frac{\partial (r\mathbf{e}_r)}{\partial \Phi} \otimes \mathbf{e}_\Phi \\ &= r'\mathbf{e}_r \otimes \mathbf{e}_R + \frac{r}{R}\left(\mathbf{e}_\theta \otimes \mathbf{e}_\Theta + \mathbf{e}_\phi \otimes \mathbf{e}_\Phi\right).\end{aligned}$$

Therefore

$$\mathbf{F} = \begin{bmatrix} r' & 0 & 0 \\ 0 & r/R & 0 \\ 0 & 0 & r/R \end{bmatrix},$$

and so

$$\lambda_r = r', \qquad \lambda_\theta = r/R, \qquad \lambda_\phi = r/R.$$

Note if we only consider *isochoric* deformation then $\det \mathbf{F} = 1$, which implies

$$r'\left(\frac{r}{R}\right)^2 = 1 \quad \Longrightarrow \quad r'r^2 = R^2 \quad \Longleftrightarrow \quad \frac{1}{3}\frac{\mathrm{d}(r^3)}{\mathrm{d}R} = R^2 \quad \Longrightarrow \quad r^3 = R^3 + C.$$

Since $r(a) = A$, $r(b) = B$,

$$C = b^3 - B^3 = a^3 - A^3 \quad \Longrightarrow \quad a^3 = b^3 - B^3 + A^3 \quad \Longrightarrow \quad r = \sqrt{a^3 - A^3 + R^3}$$

This is a one-parameter family of solutions.