LECTURE 8: INTEGRATION

The Symbolic Math Toolbox can evaluate many integrals symbolically, and MATLAB itself can also provide numerical estimates of most definite integrals.

2.5.1 The `int` operator

The syntax for integrating an expression **`expr`** with respect to a variable **`var`** is **`int(expr,var)`**. For example, to integrate the expression xe^{ax^2} with respect to x type

```
>> syms x a
>> int(x*exp(a*x^2), x)
```

Note that the Symbolic Math Toolbox does not include the arbitrary constant of integration. To evaluate the definite integral

$$\int_0^1 xe^{5x^2}\,dx$$

type

```
>> int(x*exp(5*x^2), x, 0, 1);
```

and to evaluate

$$\int_0^u \frac{dx}{\sqrt{u-x}}$$

type

```
>> syms x u
>> int(1/sqrt(u-x), x, 0, u);
```

If the Symbolic Math Toolbox cannot evaluate an integral then it returns a representation of the integral itself:

```
>> int(x^x, x)
```

which is a correct answer but probably not what you were hoping for! Tautologies like "the integral of x^x is the integral of x^x" might be somewhat frowned upon as answers in an examination.

At this point you should experiment with some integrals just to see what can be done.

Exercise 2.15 Use MATLAB to integrate the following expressions with respect to x:

(i) $\sqrt{e^x - 1}$, (ii) $x^2(ax+b)^{5/2}$, (iii) $\sinh(6x)\sinh^4(x)$, (iv) $\cosh^{-6}(x)$, (v) $\sin(\ln(x))$,

where a and b are constants. □

Exercise 2.16 Use MATLAB to evaluate the following expressions symbolically:

(i) $\displaystyle\int_{1/2}^1 \frac{1}{1+x^3}dx$, (ii) $\displaystyle\int_0^1 x^2\tan^{-1}x\,dx$, (iii) $\displaystyle\int_0^\infty \frac{1}{(1+x)(1+x^2)}\,dx$. □

From these exercises you will note that the symbolic toolbox can deal with rather nasty integrals symbolically, and that it easily evaluates many of the integrals found in elementary texts or standard tables.

Nevertheless one often encounters integrals where it is desirable to have MATLAB evaluate an integral numerically—for example, because the Symbolic Math Toolbox fails to integrate or the symbolic calculation is too slow.

2.5.2 Quadrature: numerical evaluation of integrals

You may be aware that the majority of integrals cannot be evaluated symbolically. If the symbolic toolbox fails to find a symbolic answer to a definite integral, or if you know that the integral in question cannot be evaluated in terms of known functions, then it may be necessary to evaluate it numerically.

For example, suppose we try to integrate $\int_0^1 e^{-t}\arcsin(t)dt$, symbolically by typing

```
>> syms t
>> y = exp(-t)*asin(t);
>> z = int(y, t, 0, 1)
```

As we noted before, this returns the symbolic integral expression; the Symbolic Math Toolbox does not know the answer. MATLAB has a variety of commands for doing numerical integration. This is known as *quadrature*. These are independent of the symbolic toolbox so we first convert out expression to a MATLAB function and call `integral()`. Continuing our example above:

```
>> ym = matlabFunction(y);
>> z = integral(ym, 0, 1)
```

The `integral` command has various options related to accuracy of the approximation: see `help integral`.

To summarize: at least in MATLAB R2013b, the `int()` command is part of the Symbolic Math Toolbox, and tries to evaluate integrals symbolically; the `integral()` command is a MATLAB command which evaluates integrals numerically using quadrature.

Exercise 2.17 Use MATLAB to evaluate the following integrals numerically to six digit accuracy: (i) $\displaystyle\int_0^1 e^{x^3}dx$, (ii) $\displaystyle\int_0^{10}\frac{1}{\sqrt{1+x^4}}\,dx$, (iii) $\displaystyle\int_0^5 \sin\left(e^{x/2}\right)dx$. □

Exercise 2.18 (Optional) In this exercise, we explore the function $\operatorname{sinc} x$ which is defined as

$$\operatorname{sinc} x = \begin{cases} \dfrac{\sin x}{x} & x \neq 0, \\ 1 & x = 0. \end{cases}$$

(a) Assign the expression `sin(x)/x` to a variable and then plot it for $-30 \le x \le 30$. (Note that `ezplot` should have no problem with plotting $\frac{\sin 0}{0}$. However, you should satisfy yourself that your graph looks correct at $x = 0$, given the above definition.)

(b) First, it can be shown that each maximum or minimum of the graph of $\operatorname{sinc} x$ corresponds to a point of intersection of the graphs of $\operatorname{sinc} x$ and $\cos x$. Use MATLAB to illustrate this, by drawing the graphs of $\operatorname{sinc} x$ and $\cos x$ on the same plot for $0 \le x \le 10$. Then use `solve` to find numerical approximations for the x-coordinates of all stationary points of $\operatorname{sinc} x$ for $0 < x \le 10$, and verify that these are indeed where the two graphs that you have just drawn meet.

(c) Use MATLAB to evaluate the three integrals

$$\text{(i)} \int_0^\infty \operatorname{sinc} x\,dx, \quad \text{(ii)} \int_0^\infty \operatorname{sinc} x \operatorname{sinc}\frac{x}{3}\,dx, \quad \text{(iii)} \int_0^\infty \operatorname{sinc} x \operatorname{sinc}\frac{x}{3}\operatorname{sinc}\frac{x}{5}\,dx.$$

What do you think the value of $\displaystyle\int_0^\infty \prod_{k=0}^{5}\operatorname{sinc}(x/(2k+1))\,dx$ is? Use MATLAB to verify your conjecture. □

Exercise 2.19 (Optional) In this exercise, we use MATLAB to investigate some rational bounds on π.

(a) Verify that

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx = \frac{22}{7} - \pi. \tag{2.2}$$

This is in itself an interesting result, as $\frac{22}{7}$ is often used as a rational approximation for π. However, this result can also be used to obtain bounds on π in the form $a/b < \pi < c/d$, where a, b, c and d are integers.

(b) Verify by hand, no MATLAB, that

$$\frac{1}{2}\int_0^1 x^4(1-x)^4dx < \int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx < \int_0^1 x^4(1-x)^4dx.$$

(c) Evaluate $J = \int_0^1 x^4(1-x)^4dx$, and hence deduce that

$$\frac{1979}{630} < \pi < \frac{3959}{1260}. \tag{2.3}$$

This gives rational bounds on π. You can now use MATLAB to obtain tighter rational bounds, in the following manner. It can be shown that the identity (2.2) can be generalized by replacing the powers of 4 by powers of any integer multiple of 4 as follows.

$$\int_0^1 \frac{x^{4n}(1-x)^{4n}}{2^{2(n-1)}(1+x^2)}dx = (-1)^n(\pi - R_n),$$

where $n = 1, 2, 3, \ldots$ and R_n is a rational number. (In (2.2) above, $n = 1$ and $R_1 = 22/7$.) The Symbolic Math Toolbox should be able to verify this for any given n and compute the corresponding value of R_n. Then, following the procedure of parts (ii) and (iii) above, new bounds on π can be found.[4]

(d) Use MATLAB to calculate R_5.

(e) Hence find the upper and lower bounds on π for the case when $n = 5$. Evaluate these both as rational numbers and in decimal form. You might need variable precision arithmetic (Chapter B.2). □