LECTURE 12: WATER WAVES INSTABILITY

We have always assumed thus far that the dispersion relation gives rise to *real* values of the frequency ω. However, it may well arise that ω is complex, for example when the dispersion relation is a quadratic equation such as (5.60). If we write the real and imaginary parts of ω as $\omega = \omega_R \pm i\omega_I$, then a harmonic travelling wave like (5.12) becomes

$$\begin{aligned}\eta &= A\cos(kx - \omega t - \beta) \\ &= A\cos(kx - \omega_R t - \beta)\cosh(\omega_I t) + iA\sin(kx - \omega_R t - \beta)\sinh(\omega_I t).\end{aligned} \tag{5.61}$$

We infer that a complex value of ω corresponds to an exponentially growing amplitude, and implies that the corresponding wave is *unstable.*

Example 5.2 ***Rayleigh–Taylor instability***
We return to the problem of one fluid flowing above another, analysed above in Example 5.1. *If there is no relative flow, that is $U = 0$, then the dispersion relation (5.60) reduces to*

$$\omega^2 = \frac{\left((\rho_1 - \rho_2)g + Tk^2\right)k}{\rho_1 + \rho_2}. \tag{5.62}$$

If $\rho_1 > \rho_2$ then the right-hand side of (5.62) is positive, so there are two equal and opposite values of ω, corresponding to waves propagating at speed $c = \omega/k$ in either direction. However, if $\rho_1 < \rho_2$, ω^2 is negative for some values of k, namely

$$k < \sqrt{\frac{(\rho_2 - \rho_1)g}{T}}. \tag{5.63}$$

For these wavenumbers, ω is pure imaginary, so the disturbance grows exponentially. Hence the situation with the denser fluid above the lighter fluid is (not surprisingly) unstable; this is known as the Rayleigh–Taylor instability.

Example 5.2 reminds us that the frequency ω is a function of the wavenumber k so that, in general, ω may be complex only for certain values of the wavenumber. This implies that the system is unstable only to waves of certain wavelengths. In Example 5.2, equation (5.63) implies that only waves with wavelength λ such that

$$\lambda > 2\pi\sqrt{\frac{T}{(\rho_2 - \rho_1)g}} \tag{5.64}$$

are unstable. At a water-air interface, we would have $T \approx 0.07\,\mathrm{N\,m^{-1}}$, $\rho_{\mathrm{air}} \approx 1.2\,\mathrm{kg\,m^{-3}}$, $\rho_{\mathrm{water}} \approx 1000\,\mathrm{kg\,m^{-3}}$, $g \approx 9.8\,\mathrm{N\,kg^{-1}}$, so that only waves longer than roughly 1.7 cm are unstable. If the system is too narrow to allow waves this long, then the instability will be eliminated. This explains why a glass of water may be tipped upside-down without the water spilling out if a sufficiently fine mesh is stretched over the end.

Example 5.3 ***Kelvin–Helmholtz instability***
When U is nonzero, the solution of the quadratic equation (5.60) is given by

$$\omega = \frac{\rho_2 Uk \pm \sqrt{\Delta}}{\rho_1 + \rho_2}, \tag{5.65}$$

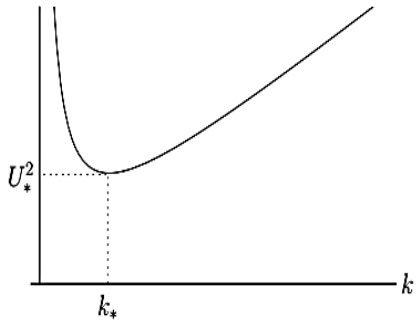

Figure 5.6: The right-hand side of (5.67) versus wavenumber k.

where Δ *is the discriminant*

$$\Delta = (\rho_1 + \rho_2)\, k\left((\rho_1 - \rho_2)g + Tk^2\right) - \rho_1\rho_2 U^2 k^2. \qquad (5.66)$$

We see that ω *is complex (so the flow is unstable) when* Δ *is negative, that is when*

$$U^2 > \left(\frac{\rho_1 + \rho_2}{\rho_1\rho_2}\right)\left(\frac{(\rho_1 - \rho_2)g}{k} + Tk\right). \qquad (5.67)$$

Assuming $\rho_1 > \rho_2$ *(so the lighter fluid is on top), the right-hand side of (5.67) tends to infinity as* $k \to 0$ *and as* $k \to \infty$*, with a minimum at* $k = k_* = \sqrt{(\rho_1 - \rho_2)g/\gamma}$*, as shown in* Figure 5.6. *This corresponds to a critical value of* U*, given by*

$$U_*^2 = \frac{2(\rho_1 + \rho_2)}{\rho_1\rho_2}\sqrt{\gamma(\rho_1 - \rho_2)g}. \qquad (5.68)$$

If $U > U_*$*, then there is a band of values of* k *for which* (5.67) *is satisfied and for which* ω *is therefore complex. In other words the flow is unstable if the velocity of the upper fluid exceeds this critical value. This* Kelvin–Helmholtz *instability is responsible for the formation of waves by wind blowing over the sea.*

5.5 Introduction to group velocity

We have seen that dispersive waves have the property that waves with different wavelengths propagate at different speeds. A localised disturbance, for example caused by dropping a pebble into a pond, will in general give rise to a spectrum of many different wavenumbers. As the waves spread out from the initial disturbance, the dispersion will cause them to be sorted according to their wavenumber. For deep water, we recall from equation (5.21) that the wave-speed c is related to the wavenumber k by $c^2 = g/k$, so that long waves travel more quickly than short waves. We would therefore expect the spreading disturbance to have longer ripples at the front and shorter ones at the back, as illustrated in Figure 5.7.

Now suppose at some time t after the initial disturbance, we detect waves with wavenumber k at a distance x from the source. We would expect these to be related through the wave-speed c by $x/t = c = \sqrt{g/k}$. However, it turns out that this prediction

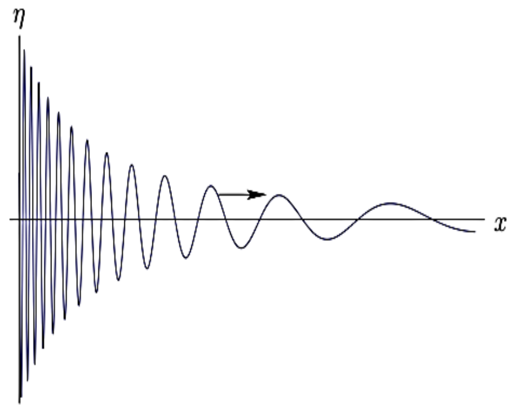

Figure 5.7: Schematic of a spreading train of ripples caused by a localised disturbance.

is out by a factor of 2. This occurs because the wave-speed c is the speed at which the crests (or troughs) propagate in a pure sinusoidal wave, not in a realistic free surface profile containing a combination of many different wavenumbers.

A profile like that illustrated in Figure 5.7 can be represented as

$$\eta(x,t) = A(x,t)\cos\big(\alpha(x,t)\big), \tag{5.69}$$

in terms of a rapidly-varying *phase* α and a slowly-varying *amplitude* A. In the vicinity of a fixed position $x = x_0$ and time $t = t_0$, we can Taylor-expand the phase to get

$$\alpha(x,t) \approx \alpha(x_0,t_0) + (x - x_0)\frac{\partial \alpha}{\partial x}(x_0,t_0) + (t - t_0)\frac{\partial \alpha}{\partial t}(x_0,t_0). \tag{5.70}$$

Hence, the profile (5.69) is locally approximated by the harmonic travelling wave

$$\eta(x,t) \approx A_0\cos\big(k_0x - \omega_0 t - \beta_0\big), \tag{5.71}$$

where

$$A_0 = A(x_0,t_0), \qquad \beta_0 = \alpha(x_0,t_0) - k_0x_0 + \omega_0 t_0, \tag{5.72a}$$

$$k_0 = \frac{\partial \alpha}{\partial x}(x_0,t_0), \qquad \omega_0 = -\frac{\partial \alpha}{\partial t}(x_0,t_0). \tag{5.72b}$$

It is therefore natural to define the local wavenumber and frequency at position x and time t by

$$k(x,t) = \frac{\partial \alpha}{\partial x}, \qquad \omega(x,t) = -\frac{\partial \alpha}{\partial t}. \tag{5.73}$$

It immediately follows from (5.73) that k and ω satisfy the equation

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0. \tag{5.74}$$

For a pure harmonic wave, there is a dispersion relation $\omega = \omega(k)$ specifying the frequency as a function of the wavenumber. We assume that the same relation holds here,

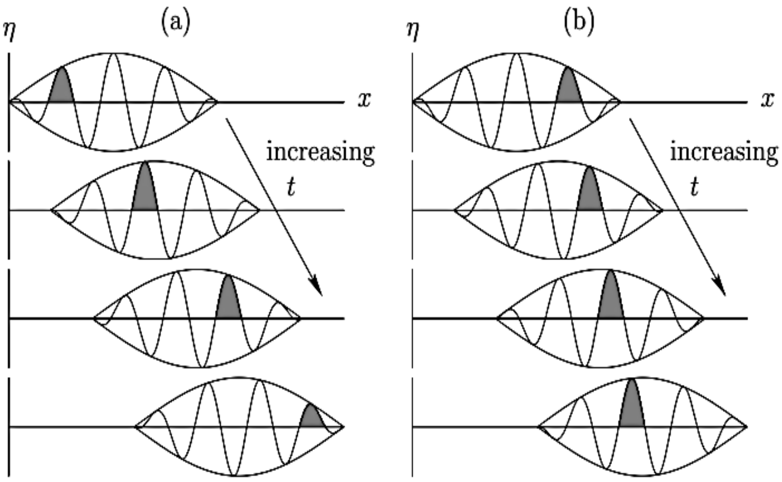

Figure 5.8: Schematic of a moving wave packet with (a) $c_g < c$, (b) $c_g > c$. One wave crest is highlighted to illustrate how it moves relative to the packet.

since the free surface profile is locally approximately sinusoidal. Thus (5.74) becomes a partial differential equation for the wavenumber k, namely

$$\frac{\partial k}{\partial t} + c_g(k)\frac{\partial k}{\partial x} = 0, \tag{5.75}$$

where

$$c_g(k) = \frac{d\omega}{dk} \tag{5.76}$$

is called the *group velocity*.

We deduce from equation (5.75) that k is constant along straight lines in the (x,t)-plane satisfying $dx/dt = c_g$. Indeed, the general solution of (5.75) is

$$k = F\big(x - c_g(k)t\big). \tag{5.77}$$

It follows that waves with wavenumber k propagate with speed $c_g(k)$, and not at the wave-speed $c(k)$ as might have been expected.

For waves on deep water, with the dispersion relation (5.21), the wave-speed and group velocity are given respectively by

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}}, \qquad c_g = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{c}{2}. \tag{5.78}$$

At first glance, this may appear to be a contradiction: how can the wave crests propagate twice as quickly as the waves themselves? The answer is that the waves separate into *wave packets* corresponding to different wavenumbers. Within each wave packet, the waves move at speed c, but the packet as a whole moves at speed c_g. This phenomenon is illustrated in Figure 5.8(a) for a single wave packet travelling from left to right at speed c_g. The wave crests move through the packet at speed $c = 2c_g$, seeming to appear at the back and disappear at the front. This behaviour can be observed in the radiating ripples caused by throwing a stone into a pond.

It is also possible for the group velocity to exceed the wave-speed; for example, it can be shown that $c_g \approx 2c$ for short capillary waves on very shallow water. If this happens, then the wave crests appear to move *backwards* relative to a radiating wave packet, as illustrated in Figure 5.8(b). This counterintuitive behaviour can sometimes be observed in small ripples on a puddle.