CSC 300-FLUID DYNAMICS & WAVES

LECTURE 11: WATER WAVES

5.1 Equations and boundary conditions

5.1.1 Setup

In this Section we will analyse so-called *Stokes waves*, namely small-amplitude waves on the free surface of an inviscid fluid, for example small ripples on a container of water. Consider fluid filling the half-space $y < 0$ with a free surface at $y = 0$, gravity acting in the $-y$-direction. Now suppose that the fluid is disturbed by small-amplitude waves, so that the free surface is displaced to $y = \eta(x, t)$, as shown schematically in Figure 5.1.

We assume that the flow is irrotational and incompressible, so that it may be described by a velocity potential ϕ such that $\boldsymbol{u} = \boldsymbol{\nabla}\phi$ and ϕ satisfies Laplace's equation. We will restrict our attention to purely two-dimensional disturbances, so that ϕ is a function of x, y and t and hence

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{5.1}$$

5.1.2 Boundary conditions

Far from the free surface, as the depth tends to infinity, we expect the velocity to tend to zero, that is

$$\boldsymbol{\nabla}\phi \to \boldsymbol{0} \quad \text{as } y \to -\infty. \tag{5.2}$$

At the free surface, there are two boundary conditions, and we will treat each separately in detail.

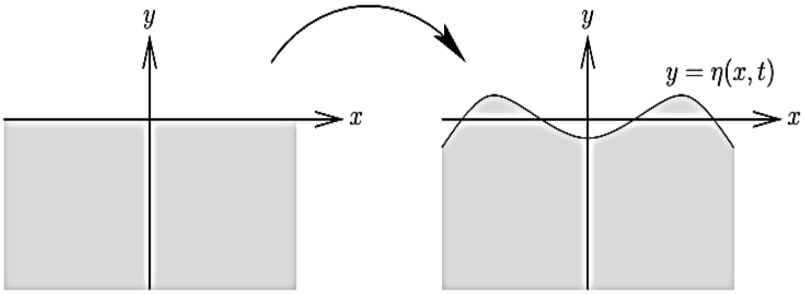
Dynamic boundary condition A force balance on the interface $y = \eta(x, t)$ implies that the pressure must be continuous there; otherwise there would be a finite force acting on a surface with zero mass, which contradicts Newton's Second Law. We therefore impose the dynamic boundary condition

$$p = P_{\text{atm}} \quad \text{at } y = \eta, \tag{5.3}$$

where P_{atm} denotes the atmospheric pressure above the fluid, which we assume to be constant.

We can write the boundary condition (5.3) in terms of the velocity potential by using Bernoulli's Theorem. For unsteady irrotational flow, we recall from Section 1 the equation

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\boldsymbol{u}|^2 + \frac{p}{\rho} + \chi = F(t), \tag{5.4}$$



(i) (ii)

Figure 5.1: (i) Fluid at rest in the half-space $y < 0$. (ii) The fluid following a disturbance that displaces the free upper surface to $y = \eta(x,t)$.

where the gravitational potential $\chi = gy$ for gravity acting in the $-y$-direction. The integration function $F(t)$ may be chosen arbitrarily by absorbing a suitable function of t into ϕ.

Evaluating (5.4) at the free surface $y = \eta$ and using (5.3), we find that

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{P_{\text{atm}}}{\rho} + g\eta = F(t) \quad \text{on } y = \eta. \tag{5.5}$$

It is convenient to choose the arbitrary function $F(t) = P_{\text{atm}}/\rho$ to cancel the constant term on the left-hand side of (5.5), and thus we obtain the dynamic boundary condition in the form

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 g\eta = 0 \quad \text{at } y = \eta. \tag{5.6}$$

Kinematic boundary condition We recall that the normal velocity of the fluid is required to be zero at a fixed impermeable wall. The corresponding condition at a moving boundary such as the free surface of a fluid is that *the velocity of the fluid normal to the boundary must equal the velocity of the boundary normal to itself.* If this were not true, the fluid would either be flowing through the boundary or separating from it, leaving behind a vacuum, neither of which is acceptable. It may be shown that this condition is equivalent to the requirement that *material fluid elements on the free surface must remain on the free surface.* Hence, if $y = \eta$ for some particular fluid particle at time t, then $y = \eta$ for the same particle for all time.

It follows that

$$\frac{\mathrm{D}}{\mathrm{D}t}(y - \eta) = 0 \quad \text{when } y - \eta = 0, \tag{5.7}$$

and, by expanding out the convective derivative, we obtain the kinematic boundary condition in the form

$$v = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} \quad \text{at } y = \eta. \tag{5.8}$$

Linearised boundary conditions Although Laplace's equation is linear, the boundary conditions (5.8) and (5.6) on the free surface are nonlinear, and the problem is therefore difficult to solve in general. If the disturbances are small, then the boundary conditions can be simplified by *linearising*, that is neglecting terms of quadratic and higher order. For example, if we neglect the quadratic terms in (5.8), we find

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} \quad \text{at } y = \eta. \tag{5.9}$$

This can be simplified further by Taylor-expanding the left-hand side as follows:

$$\frac{\partial \phi}{\partial y}(x, \eta, t) \sim \frac{\partial \phi}{\partial y}(x, 0, t) + \frac{\partial^2 \phi}{\partial y^2}(x, 0, t)\eta + \cdots, \tag{5.10}$$

in which all terms except the first are nonlinear. When linearising the boundary conditions, it is thus consistent also to evaluate the left-hand side of (5.9) at $y = 0$ rather than at $y = \eta$. The same simplification applies when we linearise (5.6), so we end up with the boundary conditions

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}, \quad \frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{at } y = 0. \tag{5.11}$$

5.2 Harmonic waves

Now we look for solutions in which the free surface displacement η takes the form of a sinusoidal travelling wave, that is

$$\eta(x, t) = A\cos(kx - \omega t - \beta), \tag{5.12}$$

where A, k, ω and β are constants. The *amplitude* of the perturbations is measured by A, while ω represents the *frequency* at which the surface oscillates at any fixed position x. The *wavenumber* k is $2\pi/\lambda$, where λ is the wavelength; thus k is small for long waves and large for short waves. The *wave-speed* at which the wave crests propagate is related to ω and k by

$$c = \frac{\omega}{k}. \tag{5.13}$$

Finally, β is an arbitrary phase shift, which may be set to zero without loss of generality by choosing the origin for t appropriately. We show a typical harmonic travelling wave in Figure 5.2.

By substituting (5.12) into the boundary conditions(5.11), we infer that ϕ is out of phase with η, so that

$$\phi(x, y, t) = f(y)\sin(kx - \omega t - \beta) \tag{5.14}$$

for some function $f(y)$ still to be determined. By substituting (5.14) into Laplace's equation (5.1), we find that $f(y)$ satisfies the ordinary differential equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}y^2} - k^2 f = 0. \tag{5.15}$$

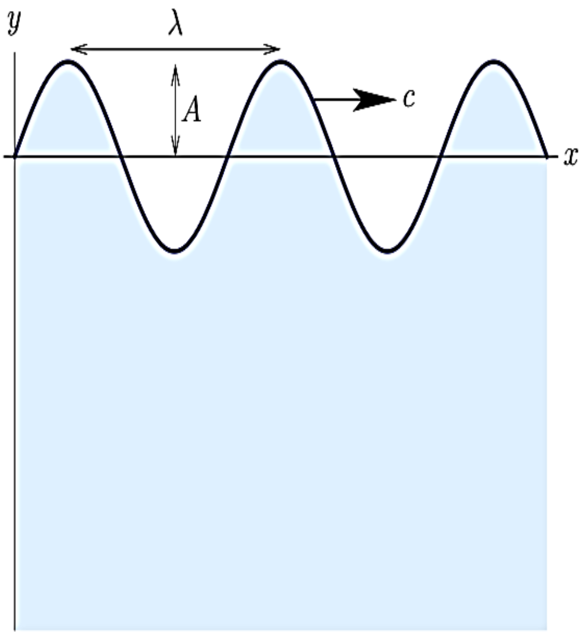


Figure 5.2: Schematic of a harmonic travelling wave, showing the amplitude A, wavelength λ and wave-speed c.

The far-field condition (5.2) and the free-surface conditions (5.11) imply that $f(y)$ must satisfy the boundary conditions

$$f(y) \to 0 \quad \text{as } y \to -\infty, \tag{5.16}$$

$$f'(0) = \omega A, \quad -\omega f(0) + gA = 0. \tag{5.17}$$

Without loss of generality, we assume that k is positive, so the solution of (5.15) that satisfies the far-field condition (5.16) is

$$f(y) = Be^{ky} \tag{5.18}$$

for some constant B. The boundary conditions (5.17) at $y = 0$ thus give us a system of linear equations for the two constants A and B, which may be written in the form

$$\begin{pmatrix} \omega & -k \\ g & -\omega \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.19}$$

The homogeneous linear system (5.19) admits the trivial solution $A = B = 0$, corresponding to η and ϕ both being identically zero. A nontrivial solution can only exist if the determinant of the left-hand side is zero, that is if

$$\omega^2 = gk. \tag{5.20}$$

This equation for the frequency in terms of the wavenumber is called the *dispersion relation.* The corresponding wave-speed c satisfies

$$c^2 = \frac{g}{k}, \tag{5.21}$$

which depends on the wavenumber k, so that waves with different wavenumbers move at different speeds. Such waves are called *dispersive*, in contrast with waves on a string or sound waves, for example, which have a constant wave speed.

We see from (5.21) that the wave-speed is a decreasing function of the wavenumber, so that longer waves propagate more quickly. In principle, the wave-speed may be arbitrarily large for very long waves. We will see below that this is an artefact of our assumption that the fluid has infinite depth.

5.3 Generalisations

5.3.1 Finite depth

The analysis performed above is easily generalised to describe waves on a fluid of finite depth h. Suppose fluid occupies the region $-h < y < \eta(x,t)$ between a rigid base at $y = -h$ and a free surface at $y = \eta(x,t)$. We recall that the normal velocity at the base must be zero, and hence ϕ must satisfy the boundary condition

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{at } y = -h. \tag{5.22}$$

This replaces the far-field condition (5.2); otherwise the problem is identical to that solved in §5.2.

We again seek a solution in the form of a harmonic travelling wave, so that

$$\eta(x,t) = A\cos(kx - \omega t - \beta), \qquad \phi(x,y,t) = f(y)\sin(kx - \omega t - \beta), \tag{5.23}$$

for some function $f(y)$. By substituting this expression for ϕ into Laplace's equation, we again find that $f(y)$ satisfies the differential equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}y^2} - k^2 f = 0 \tag{5.24}$$

and the boundary conditions

$$f'(0) = \omega A, \quad -\omega f(0) + gA = 0. \tag{5.25}$$

However, the condition (5.22) on the base now leads to the boundary condition

$$f'(-h) = 0. \tag{5.26}$$

Clearly the general solution of (5.24) is a linear combination of e^{ky} and e^{-ky}. Alternatively, we can write $f(y)$ as a combination of $\cosh(ky)$ and $\sinh(ky)$. However, the neatest approach is to note that the boundary condition (5.26) is satisfied identically by setting

$$f(y) = B\cosh\big(k(y+h)\big), \tag{5.27}$$

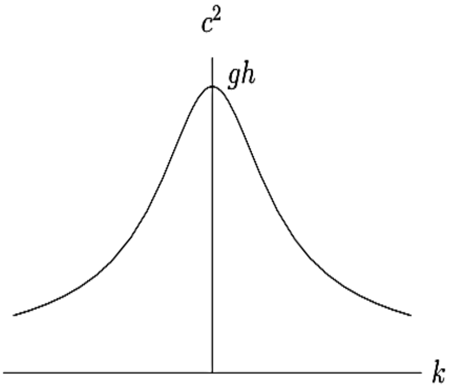


Figure 5.3: The squared wave-speed c^2 given by (5.30) versus wavenumber k.

for some constant B. Substitution into the free-surface conditions (5.25) again leads to a system of linear equations for A and B, which now takes the form

$$\begin{pmatrix} \omega & -k\sinh(kh) \\ g & -\omega\cosh(kh) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.28}$$

For nontrivial solutions, the determinant of the system must be zero, and this now gives us the dispersion relation

$$\omega^2 = gk\tanh(kh). \tag{5.29}$$

The wave-speed c is therefore given by

$$c^2 = \frac{g}{k}\tanh(kh). \tag{5.30}$$

As depicted in figure 5.3, for positive k, the right-hand side of (5.30) is a decreasing function, indicating that long waves travel faster than short waves. However, the wave-speed is now bounded, with a maximum achieved in the limit $k \to 0$, where we find that

$$c \to \sqrt{gh} \quad \text{as } k \to 0. \tag{5.31}$$

5.3.2 Flowing fluid

We can study waves on a flowing liquid by linearising about uniform flow, setting $\boldsymbol{u} = U\boldsymbol{i} + \boldsymbol{\nabla}\phi$, that is,

$$u = U + \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}, \tag{5.32}$$

where ϕ and its derivatives are again assumed to be small. It is clear that ϕ still satisfies Laplace's equation. Furthermore, if we consider fluid of finite depth h, with a rigid impermeable base at $y = -h$, then ϕ still satisfies the boundary condition

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{at } y = -h. \tag{5.33}$$

At the free surface, the kinematic boundary condition (5.8) now reads

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \left(U + \frac{\partial \phi}{\partial x}\right)\frac{\partial \eta}{\partial x} \quad \text{at } y = \eta(x,t). \tag{5.34}$$

When we linearise, as in §5.1.2, this is simplified to

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} \quad \text{at } y = 0. \tag{5.35}$$

Next we turn to the dynamic boundary condition. With the velocity given by (5.32), Bernoulli's equation (5.4) is modified to

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|U\boldsymbol{i} + \boldsymbol{\nabla}\phi|^2 + \frac{p}{\rho} + gy = F(t). \tag{5.36}$$

Setting p equal to the atmospheric pressure P_{atm} at the free surface $y = \eta$, we therefore obtain the boundary condition

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(U + \frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \frac{P_{\text{atm}}}{\rho} + g\eta = F(t) \quad \text{at } y = \eta(x,t). \tag{5.37}$$

It is convenient to choose the arbitrary function $F(t)$ to cancel the constant terms on the left-hand side, that is

$$F(t) = \frac{1}{2}U^2 + \frac{P_{\text{atm}}}{\rho}. \tag{5.38}$$

Then linearisation of (5.37) leads to the condition

$$\frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} + g\eta = 0 \quad \text{at } y = 0. \tag{5.39}$$

Again, we can seek travelling-wave solutions of the form (5.23). The modified boundary conditions (5.35) and (5.39) imply that $f(y)$ must satisfy

$$f'(0) = (\omega - Uk)A, \quad -(\omega - Uk)f(0) + gA = 0. \tag{5.40}$$

The boundary condition (5.33) at the base again implies that $f(y)$ should take the form

$$f(y) = B\cosh\big(k(y+h)\big), \tag{5.41}$$

and substitution into (5.40) leads to the homogeneous linear system

$$\begin{pmatrix} \omega - Uk & -k\sinh(kh) \\ g & -(\omega - Uk)\cosh(kh) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.42}$$

For there to exist nontrivial solutions, ω must satisfy the dispersion

$$(\omega - Uk)^2 = gk\tanh(kh). \tag{5.43}$$

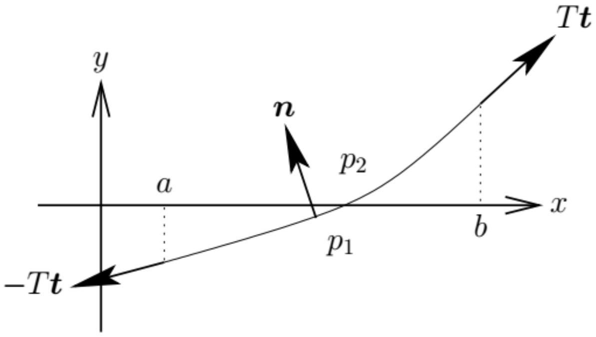


Figure 5.4: Schematic showing the surface tension T acting at a fluid interface.

We deduce that there are two possible wave-speeds, namely

$$c_\pm = U \pm \left(\frac{g}{k}\tanh(kh)\right)^{1/2}. \tag{5.44}$$

We recall that the bracketed term in this equation is bounded by gh, as shown in Figure 5.3. Hence we can identify two possible cases in (5.44). If the flow speed U is less than $\sqrt{gh}$, then waves may propagate both upstream and downstream. Such a flow is termed *subcritical*. On the other hand, if $U > \sqrt{gh}$, then all waves are carried downstream and the flow is said to be *supercritical*.

5.3.3 Two fluids

Now suppose the interface $y = \eta$ separates two fluids with different densities, say $\rho = \rho_1$ in $y < 0$ and $\rho = \rho_2$ in $y > 0$. We denote the velocity potentials and pressures on either side by ϕ_1, ϕ_2 and p_1, p_2 respectively. The kinematic condition (5.8) applies to both sides of the interface, and leads to the linearised boundary conditions

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi_1}{\partial y} = \frac{\partial \phi_2}{\partial y} \quad \text{at } y = 0. \tag{5.45}$$

The dynamic boundary condition (5.3) is replaced by the pressure continuity condition $p_1 = p_2$ at $y = \eta$. After use of Bernoulli's equation (5.4) and linearisation, this leads to the boundary condition

$$\rho_1\left(\frac{\partial \phi_1}{\partial t} + g\eta\right) = \rho_2\left(\frac{\partial \phi_2}{\partial t} + g\eta\right) \quad \text{at } y = 0. \tag{5.46}$$

Notice that (5.11) is recovered if we let the density ratio ρ_2/ρ_1 tend to zero.

5.3.4 Surface tension

Real fluid interfaces exhibit a phenomenon called *surface tension*, which acts like a membrane stretched over the interface to a tension T. In Figure 5.4 we show the forces acting on small element of the interface between $x = a$ and $x = b$, namely the pressures on either side and the surface tension at the ends. These forces must sum to zero, that is

$$\int_{x=a}^{x=b} (p_1 - p_2)\boldsymbol{n}\,\mathrm{d}s + \big[T\boldsymbol{t}\big]_{x=a}^{x=b} = \boldsymbol{0}, \tag{5.47}$$

where $\mathrm{d}s$ denotes integration with respect to arc length, $\boldsymbol{t}$ is the unit tangent and $\boldsymbol{n}$ is the unit normal to the interface, chosen to point from fluid 1 into fluid 2, as shown in Figure 5.4. These are given respectively by

$$\mathrm{d}s = \sqrt{1+\eta_x^2}\,\mathrm{d}x, \qquad \boldsymbol{t} = \frac{1}{\sqrt{1+\eta_x^2}}\begin{pmatrix}1\\ \eta_x\end{pmatrix}, \qquad \boldsymbol{n} = \frac{1}{\sqrt{1+\eta_x^2}}\begin{pmatrix}-\eta_x\\ 1\end{pmatrix}. \tag{5.48}$$

By using the Fundamental Theorem of Calculus, we can write (5.47) in the form

$$\int_a^b \left((p_1 - p_2)\boldsymbol{n}\sqrt{1+\eta_x^2} + \frac{\partial}{\partial x}(T\boldsymbol{t})\right)\mathrm{d}x = \boldsymbol{0}. \tag{5.49}$$

This must be true for all intervals $[a, b]$ along the surface, and the integrand, if continuous, must therefore be zero. The surface tension T is assumed to be constant, and we therefore obtain the boundary condition

$$(p_1 - p_2)\boldsymbol{n} + \frac{T}{\sqrt{1+\eta_x^2}}\frac{\partial \boldsymbol{t}}{\partial x} = \boldsymbol{0} \quad \text{at } y = \eta. \tag{5.50}$$

Direct differentiation of (5.48) reveals that

$$\frac{1}{\sqrt{1+\eta_x^2}}\frac{\partial \boldsymbol{t}}{\partial x} = \kappa \boldsymbol{n}, \tag{5.51}$$

where

$$\kappa = \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} \tag{5.52}$$

is the *curvature* of the interface. Hence we deduce from (5.50) that there is a pressure jump across the interface equal to

$$p_2 - p_1 = T\kappa \quad \text{at } y = \eta. \tag{5.53}$$

After linearisation, this reads

$$p_2 - p_1 = T\eta_{xx} \quad \text{at } y = 0, \tag{5.54}$$

and the dynamic boundary condition (5.46) is thus modified to

$$\rho_1\left(\frac{\partial \phi_1}{\partial t} + g\eta\right) - \rho_2\left(\frac{\partial \phi_2}{\partial t} + g\eta\right) = T\frac{\partial^2 \eta}{\partial x^2} \quad \text{at } y = 0 \tag{5.55}$$

to take account of surface tension. Note that (5.46) is recovered if T is set to zero.

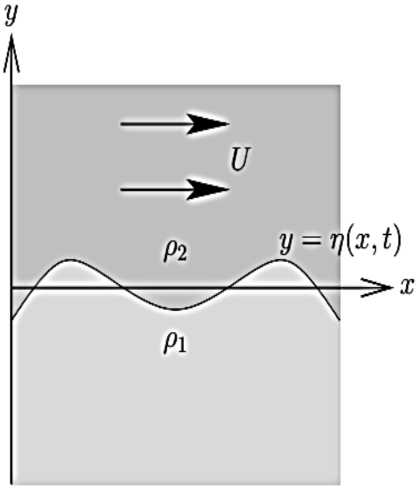


Figure 5.5: Schematic of a fluid of density ρ_2 flowing at speed U over a fluid of density ρ_1.

Example 5.1 ***One fluid flowing over another***

We illustrate the effects outlined above by analysing the situation shown in Figure 5.5, *where an infinite layer of fluid with density ρ_2 flows at speed U over an infinite layer of density ρ_1. We include a surface tension T at the interface between the two fluids, so that the disturbance potentials ϕ_1, ϕ_2 and the free-surface deflection η satisfy*

$$\nabla^2\phi_1 = 0 \qquad \text{in } y < 0, \qquad \nabla^2\phi_2 = 0 \qquad \text{in } y > 0, \tag{5.56}$$

$$\left.\begin{aligned} \frac{\partial\phi_1}{\partial y} = \frac{\partial\eta}{\partial t}, \qquad \frac{\partial\phi_2}{\partial y} = \frac{\partial\eta}{\partial t} + U\frac{\partial\eta}{\partial x}, \\ \rho_1\left(\frac{\partial\phi_1}{\partial t} + g\eta\right) - \rho_2\left(\frac{\partial\phi_2}{\partial t} + U\frac{\partial\phi_2}{\partial x} + g\eta\right) = T\frac{\partial^2\eta}{\partial x^2} \end{aligned}\right\} \quad \text{at } y = 0. \tag{5.57}$$

As usual, we look for harmonic travelling waves with $\eta(x,t) = A\cos(kx - \omega t - \beta)$, where the wavenumber k is assumed to be positive, without loss of generality. The corresponding solutions ϕ_1, ϕ_2 of Laplace's equation that decay as $y \to -\infty$ and $y \to +\infty$ respectively are then easily found to be

$$\phi_1(x,y,t) = B\mathrm{e}^{ky}\sin(kx - \omega t - \beta), \qquad \phi_2(x,y,t) = C\mathrm{e}^{-ky}\sin(kx - \omega t - \beta), \tag{5.58}$$

where B and C are arbitrary constants.

On substituting these into the boundary conditions (5.57)*, we obtain a system of three homogenous linear equations for A, B and C, which can be written in the form*

$$\begin{pmatrix} \omega & -k & 0 \\ \omega - Uk & 0 & k \\ (\rho_1 - \rho_2)g + Tk^2 & -\rho_1\omega & \rho_2(\omega - Uk) \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.59}$$

For nontrivial solutions, the determinant of the system must be zero, and hence we obtain the dispersion relation

$$(\rho_1 + \rho_2)\omega^2 - 2(\rho_2 Uk)\omega + \rho_2 U^2k^2 - (\rho_1 - \rho_2)gk - Tk^3 = 0. \tag{5.60}$$