LECTURE 6: CIRCLE THEOREM AND APPLICATION

Theorem 2.1 ***Milne-Thomson's Circle Theorem***
Suppose a velocity potential $w(z) = f(z)$ is given such that any singularities in $f(z)$ occur in $|z| > a$. Then the potential

$$w(z) = f(z) + \overline{f\left(\frac{a^2}{\overline{z}}\right)} \tag{2.71}$$

(where the bar again denotes complex conjugate) has

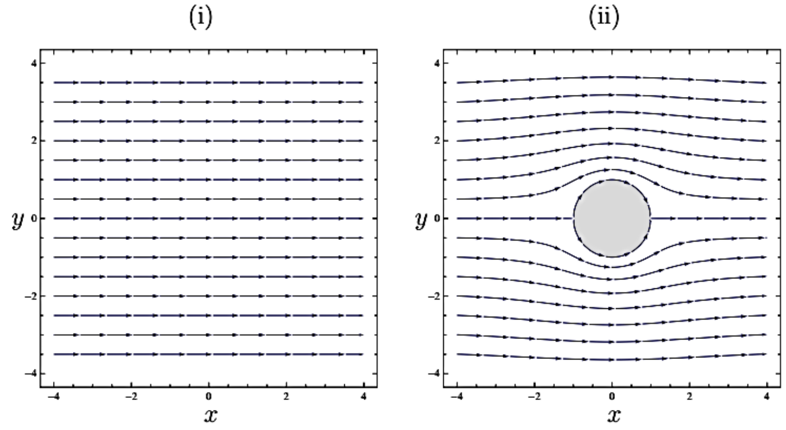


Figure 2.13: (i) Uniform flow in the x-direction. (ii) Uniform flow past a circular cylinder.

1. *the same singularities as $f(z)$ in $|z| > a$;*
2. *the circle $|z| = a$ as a streamline.*

Proof To prove part 1, we note that, if $|z| > a$, then $|a^2/\overline{z}| < a$. Since $f(z)$ is assumed to have no singularities in $|z| \leqslant a$, it follows that the second term in (2.71) has no singularities in $|z| > a$ and the result follows.

For part 2, note that, when $|z| = a$, we have $\overline{z} = a^2/z$ and hence

$$w(z)\big|_{|z|=a} = f(z) + \overline{f(z)} = 2\,\mathrm{Re}\big(f(z)\big).$$

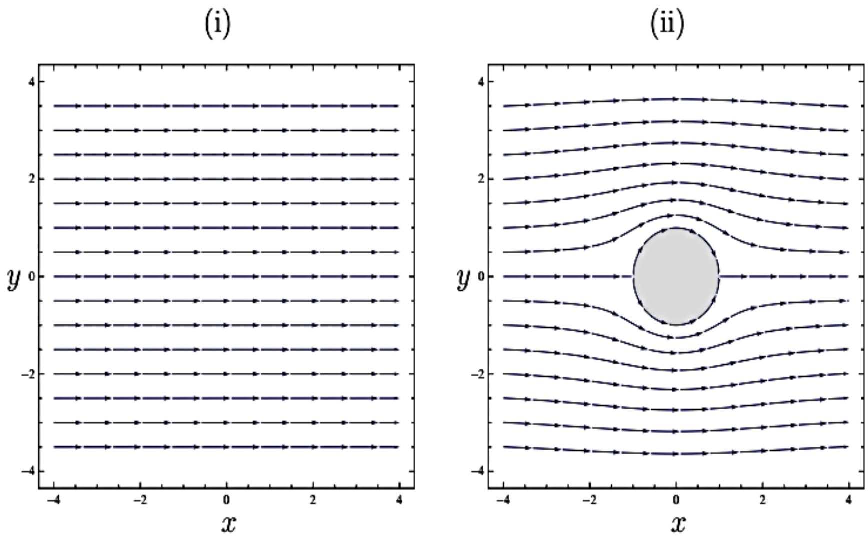
The right-hand side is evidently real, and it follows that $\mathrm{Im}\, w(z) = \psi = 0$ on $|z| = a$, and the circle $|z| = a$ is therefore a streamline. ∎

This theorem allows us to find the resulting flow when a circular cylinder is placed in a background flow with complex potential $f(z)$.

Example 2.15 ***Uniform flow past a circular cylinder***
Our starting point is uniform flow at speed U in the x-direction, as shown in Figure 2.13(i), *with complex potential $w(z) = Uz$. If we insert an impermeable circular obstacle, at $|z| = a$ say, then the flow will be disturbed as illustrated in* Figure 2.13(ii). *Our task is to calculate the disturbed flow.*

Note that $f(z) = Uz$ has a pole at infinity but is holomorphic in the disc $|z| \leqslant a$ and thus satisfies the hypotheses of the Circle Theorem. We can therefore obtain the solution by substituting $f(z) = Uz$ into equation (2.71):

$$w(z) = Uz + \frac{\overline{Ua^2}}{\overline{z}} \equiv Uz + \frac{Ua^2}{z}. \tag{2.72}$$



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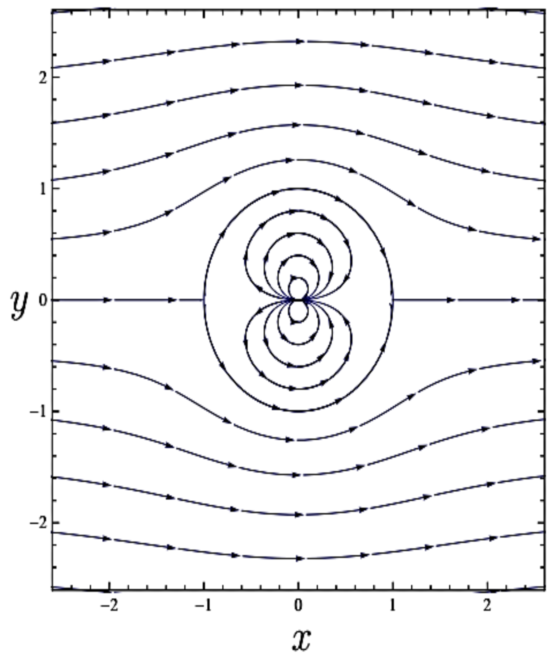

Figure 2.14: Streamlines caused by a doublet in a uniform flow.

The streamfunction is just the imaginary part of this function, namely

$$\psi = Uy\left(1 - \frac{a^2}{x^2+y^2}\right), \tag{2.73}$$

and we see that the circle $x^2 + y^2 = a^2$ is indeed a streamline, with $\psi = 0$. The resulting flow (with $a = 1$) is shown in Figure 2.13(ii).

The Circle Theorem may be viewed as an extension of the method of images to circular boundaries. In Example 2.15, we see that application of the Circle Theorem results in a singularity being inserted inside the cylinder. The flow due to the "image" singularity then repels the uniform flow, such that the circle $|z| = a$ ends up being a streamline. In this case, the required singularity turns out to be a so-called *doublet*, with complex potential $w(z) = Ua^2/z$. The singularity at the origin is inside the obstacle and thus does not adversely affect the external flow. The full streamline pattern, including the doublet inside the cylinder, is shown in Figure 2.14.

The Circle Theorem allows us to construct a velocity potential that shares the same singularities with the given external flow $f(z)$ and has the circle $|z| = a$ as a streamline. However, it does not guarantee that the resulting potential is unique. Consider the flow due to a *vortex*, as illustrated in Figure 2.4. This is a velocity field that has $|z| = a$ as a streamline, has no singularities in $|z| > a$ and decays to zero as $|z| \to \infty$. Hence, we could superimpose the flow due to a vortex at the origin onto any flow outside a circular cylinder, without introducing any further singularities in the fluid or disturbing the streamline at $|z| = a$. The most general flow produced by a inserting a circular

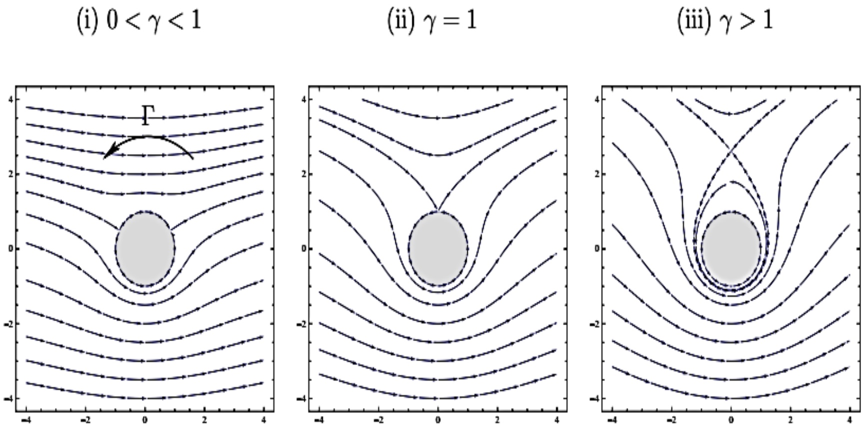

Figure 2.15: Streamlines for uniform flow at speed U flow past a cylinder pf radius a with with circulation Γ. (i) $0 < \gamma < 1$; (ii) $\gamma = 1$; (iii) $\gamma > 1$, where $\gamma = \Gamma/4\pi U a$.

cylinder into an external flow $w(z) = f(z)$ is thus

$$w(z) = f(z) + \overline{f\left(\frac{a^2}{\overline{z}}\right)} - \frac{\mathrm{i}\Gamma}{2\pi}\log z, \tag{2.74}$$

where Γ is arbitrary, and represents the *circulation* about the cylinder.

Example 2.16 ***Uniform flow past a circular cylinder with circulation***
If we incorporate circulation in the complex potential (2.72) *corresponding to uniform flow past a circular cylinder, we obtain*

$$w(z) = Uz + \frac{Ua^2}{z} - \frac{\mathrm{i}\Gamma}{2\pi}\log z. \tag{2.75}$$

We can read off the streamfunction from the imaginary part of this function, namely

$$\psi = Uy\left(1 - \frac{a^2}{x^2+y^2}\right) - \frac{\Gamma}{4\pi}\log\left(x^2+y^2\right). \tag{2.76}$$

Hence the circle $x^2 + y^2 = a^2$ *is still a streamline, with* $\psi = -(\Gamma/2\pi)\log a$.

Any stagnation points in the flow satisfy

$$0 = u - \mathrm{i}v = \frac{\mathrm{d}w}{\mathrm{d}z} = U - \frac{Ua^2}{z^2} - \frac{\mathrm{i}\Gamma}{2\pi z}, \tag{2.77}$$

which can be rearranged to the quadratic equation

$$z^2 - 2\mathrm{i}\gamma a z - a^2 = 0, \tag{2.78}$$

where for convenience we introduce the shorthand

$$\gamma = \frac{\Gamma}{4\pi U a}. \tag{2.79}$$

The roots of equation (2.78) *are given by*

$$\frac{z}{a} = \mathrm{i}\gamma \pm \sqrt{1-\gamma^2}. \tag{2.80}$$

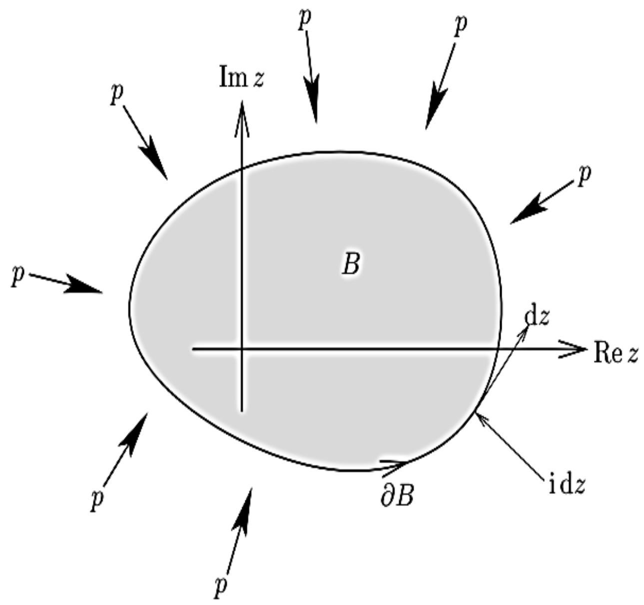

Figure 2.16: Schematic of an obstacle B, with boundary ∂B, subject to a pressure p.

When $\gamma = 0$, there is no circulation and the flow is as illustrated in Figure 2.13(ii), *with stagnation points at $z = \pm a$. As γ increases, the anticlockwise circulation causes the stagnation points to move upwards around the cylinder, as shown in* Figure 2.15(i). *When γ reaches the value 1, the two stagnation points coalesce at the top of the cylinder $z = \mathrm{i}a$, as shown in* Figure 2.15(ii). *If $\gamma > 1$, then one stagnation point moves into the flow, as shown in* Figure 2.15(iii)*; the other one is inside the cylinder. Now there is a region of fluid, bounded by the separatrix through the stagnation point, that simply rotates around the cylinder, and is completely cut off from the external flow.*

2.4 Blasius' Theorem

Suppose we know the velocity potential $w(z)$ corresponding to flow past a given obstacle B, and we wish to determine the *force* exerted on the obstacle by the fluid flowing past. We will show how to achieve this for an arbitrarily-shaped obstacle. Although so far we have learnt how to determine $w(z)$ only when B is circular, we will find that the Circle Theorem may be generalised to other shapes by using conformal mapping.

Lemma 2.2 *Suppose fluid flows steadily past an obstacle B with simple closed boundary ∂B. If gravity is neglected, the net force (F_x, F_y) exerted on B by the fluid (per unit length out of the plane) is given by*

$$F_x + \mathrm{i}F_y = -\frac{\mathrm{i}\rho}{2} \oint_{\partial B} \left| \frac{\mathrm{d}w}{\mathrm{d}z} \right|^2 \mathrm{d}\bar{z}. \tag{2.81}$$

Proof We recall that the pressure is assumed to act in the inward normal direction, so the net force experienced by the body B is given by

$$(F_x, F_y) = \oint_{\partial B} -p\boldsymbol{n}\,\mathrm{d}s = \oint_{\partial B} p(-\mathrm{d}y, \mathrm{d}x). \tag{2.82}$$

We translate the components (F_x, F_y) of $\boldsymbol{F}$ into a complex scalar by writing

$$F_x + \mathrm{i}F_y = \oint_{\partial B} p(-\mathrm{d}y + \mathrm{i}\mathrm{d}x) = \oint_{\partial B} p\mathrm{i}\,\mathrm{d}z, \tag{2.83}$$

where $\mathrm{d}z = \mathrm{d}x + \mathrm{i}\mathrm{d}y$. The directions of $\mathrm{d}z$ and $\mathrm{i}\,\mathrm{d}z$ are illustrated in Figure 2.16.

To evaluate the pressure from the complex potential, we turn to Bernoulli's Theorem. Restricting our attention to steady flow and neglecting gravity, we substitute the velocity components from (2.46) into (2.9) to obtain

$$p = P - \frac{\rho}{2}\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2, \tag{2.84}$$

where P is a constant reference pressure. Equation (2.81) follows from substitution of (2.84) into (2.83); the constant term P in the pressure integrates to zero by Cauchy's Theorem. $\blacksquare$

Although equation (2.81) gives us formulae for the force components, it is very inconvenient to use it as it stands. The modulus signs mean that the integrand in (2.81) will in general be holomorphic nowhere. Hence, none of the standard tools of complex analysis is any use in evaluating the right-hand side of (2.81). Fortunately, there is a neat trick that eliminates the modulus signs in (2.81) and thus transforms the integrand into a meromorphic function.

Theorem 2.3 ***Blasius' Theorem***
Suppose fluid flows steadily past an obstacle B with simple closed boundary ∂B. If gravity is neglected, the net force (F_x, F_y) exerted on B by the fluid is given by

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}\rho}{2}\oint_{\partial B}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z. \tag{2.85}$$

Proof By taking the complex conjugate of (2.81), we obtain

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}\rho}{2}\oint_{\partial B}\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 \mathrm{d}\bar{z}, \tag{2.86}$$

where $\mathrm{d}\bar{z} = \mathrm{d}x - \mathrm{i}\mathrm{d}y$. By factorising the integrand in the form

$$\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 \equiv \frac{\mathrm{d}w}{\mathrm{d}z}\,\frac{\mathrm{d}\overline{w}}{\mathrm{d}\bar{z}}, \tag{2.87}$$

we can rewrite (2.86) as

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}\rho}{2}\oint_{\partial B}\frac{\mathrm{d}w}{\mathrm{d}z}\,\frac{\mathrm{d}\overline{w}}{\mathrm{d}\bar{z}}\,\mathrm{d}\bar{z} \equiv \frac{\mathrm{i}\rho}{2}\oint_{\partial B}\frac{\mathrm{d}w}{\mathrm{d}z}\,\mathrm{d}\overline{w}. \tag{2.88}$$

Now we recall that ∂B is supposed to be a streamline for the flow. Hence $\psi = \operatorname{Im} w$ is constant on ∂B and it follows that $\mathrm{d}\overline{w} \equiv \mathrm{d}w$ on ∂B. We therefore obtain

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}\rho}{2} \oint_{\partial B} \frac{\mathrm{d}w}{\mathrm{d}z}\,\mathrm{d}w, \tag{2.89}$$

and equation (2.85) follows. ∎

(For an alternative proof of Blasius' Theorem, see Acheson, section 4.10.)

Example 2.17 ***Uniform flow past a circular cylinder***
We illustrate the use of Blasius' Theorem for the flow considered in Example 2.15. *Substituting the velocity potential* (2.72), *into equation* (2.85), *we find the force components*

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}\rho}{2} \oint_{|z|=a} \left(U - \frac{Ua^2}{z^2}\right)^2 \mathrm{d}z. \tag{2.90}$$

This integral is easily evaluated using Cauchy's Residue Theorem. *There is just one pole, of order* 4, *at the origin. However, the Laurent expansion of the integrand, namely*

$$\frac{U^2a^4}{z^4} - \frac{2Ua^2}{z^2} + U^2$$

has no term of order $1/z$, *and the residue of the pole is therefore zero.*

Example 2.17 shows that *the net force exerted on a circular cylinder by a uniform flow is zero!* This counter-intuitive result is known as *D'Alembert's Paradox.* In practice, we would certainly expect a cylinder subject to a uniform flow to experience a nonzero force. The discrepancy between the theoretical prediction and experimental reality is now known to be due to our neglect of viscous effects.

Example 2.18 ***Uniform flow past a circular cylinder with circulation***
Next we apply Blasius' Theorem to the flow considered in Example 2.16, *with circulation included. Substituting the velocity potential* (2.75), *into equation* (2.85), *we find the force components*

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}\rho}{2} \oint_{|z|=a} \left(U - \frac{Ua^2}{z^2} - \frac{\mathrm{i}\Gamma}{2\pi z}\right)^2 \mathrm{d}z. \tag{2.91}$$

Again, there is just one pole, of order 4, at the origin. When we expand out the integrand, we easily find that the coefficient of $1/z$ *is* $-\mathrm{i}U\Gamma/\pi$, *and we therefore deduce from Cauchy's Residue Theorem that*

$$F_x - \mathrm{i}F_y = \mathrm{i}\rho U\Gamma. \tag{2.92}$$

Hence the so-called drag force F_x, *parallel to the flow, is zero, while the* lift force, *transverse to the flow, is given by*

$$F_y = -\rho U\Gamma. \tag{2.93}$$

We can interpret the lift formula (2.93) as follows. If Γ is positive (as in Figure 2.15), then the circulation opposes the uniform flow above the cylinder and reinforces it below. Hence the average speed along the bottom of the cylinder will be higher than that along the top. Bernoulli's Theorem tells us that higher speed is associated with lower pressure,