LECTURE 5: COMPLEX POTENTIAL FLOW

The velocity potential and streamfunction give us the two alternative representations (2.4) and (2.22) of the velocity. By comparing these expressions, we see that ϕ and ψ must satisfy the equations

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{2.43}$$

We recognise these as the *Cauchy–Riemann equations*, which relate the real and imaginary parts of a holomorphic function. Provided ϕ and ψ are continuously differentiable, we can deduce from (2.43) that

$$\phi(x, y) + \mathrm{i}\psi(x, y) \equiv w(z), \tag{2.44}$$

where w is a holomorphic function of $z = x + \mathrm{i}y$, known as the *complex potential.* We can use many of the powerful techniques of complex analysis to determine and manipulate the complex potential, and this allows us to solve for ϕ and ψ simultaneously.

We recall that the holomorphic function $w(z)$ has a well-defined derivative which is independent of the direction of differentiation in the complex plane, that is

$$\frac{\mathrm{d}w}{\mathrm{d}z} \equiv \frac{\partial}{\partial x}(\phi + \mathrm{i}\psi) \equiv \frac{1}{\mathrm{i}}\frac{\partial}{\partial y}(\phi + \mathrm{i}\psi). \tag{2.45}$$

Hence we find that the velocity components (u, v) may be recovered from w using the formula

$$\frac{\mathrm{d}w}{\mathrm{d}z} \equiv u - \mathrm{i}v. \tag{2.46}$$

We illustrate the use of $w(z)$ by listing the complex potentials corresponding to each of the examples introduced in §2.1.

Example 2.10 ***Uniform flow***

The complex potential

$$w(z) = U\mathrm{e}^{-\mathrm{i}\alpha}z \tag{2.47}$$

corresponds to uniform flow at speed U in a direction making an angle α with the x-axis. We can easily see this by calculating $\mathrm{d}w/\mathrm{d}z$ *and substituting into* (2.46).

Example 2.11 ***Stagnation point flow***

The velocity field $(u, v) = (x, -y)$ *corresponds to the complex potential*

$$w(z) = \frac{z^2}{2}. \tag{2.48}$$

We can easily see that the real and imaginary parts agree with equations (2.13) *and* (2.33).

Example 2.12 ***Line source***

A line source of strength Q at the origin is represented by the complex potential

$$w(z) = \frac{Q}{2\pi}\log z. \tag{2.49}$$

We note that this is a multifunction, *with a branch point at the origin. This is no surprise when we recall that a line source has a multivalued streamfunction.*

It is easy to generalise (2.49) to a source at an arbitrary point, say (a, b), in the (x, y)-plane. The required complex potential is

$$w(z) = \frac{Q}{2\pi}\log(z - c), \tag{2.50}$$

where $c = a + \mathrm{i}b$ is the complex number corresponding to the point (a, b) in the complex plane.

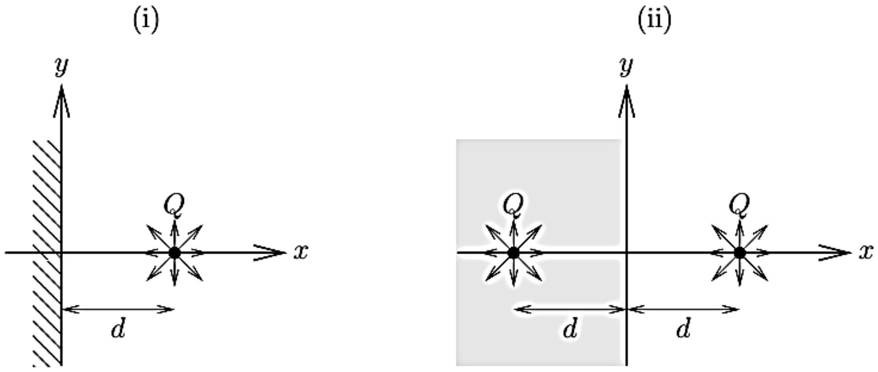

Figure 2.5: (i) Schematic of a source in the half-plane $x > 0$ bounded by a wall at $x = 0$. (ii) The same flow mimicked by inserting an image source.

Example 2.13 ***Line vortex***
A line vortex of strength Γ at the origin is represented by the complex potential

$$w(z) = \frac{-\mathrm{i}\Gamma}{2\pi} \log z. \tag{2.51}$$

This again is a multifunction, and we can infer from (2.51) *that the velocity potential is the multivalued function*

$$\phi = \frac{\Gamma}{2\pi}\theta. \tag{2.52}$$

The jump in ϕ that occurs upon a circuit of the origin corresponds to the circulation Γ.

A vortex at an arbitrary point (a, b) is represented by the complex potential

$$w(z) = \frac{-\mathrm{i}\Gamma}{2\pi} \log(z - c), \tag{2.53}$$

where $c = a + \mathrm{i}b$.

These basic flows can all be combined by simply superimposing the corresponding complex potentials.

2.3 Method of images

2.3.1 Plane boundaries

Now we consider how some of the flows that we have considered so far are modified in the presence of boundaries. As a first example, suppose fluid occupies the half-space $x > 0$ and there is a rigid impermeable wall along the y-axis $x = 0$. Now we wish to find the flow caused by a source of strength Q placed in the fluid at the point $(d, 0)$, where $d > 0$. We have to impose the boundary condition of zero normal velocity through the wall, that is

$$u = 0 \quad \text{at } x = 0. \tag{2.54}$$

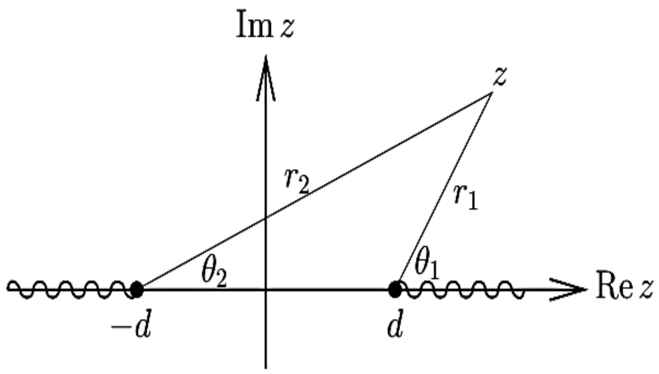

Figure 2.6: The lengths r_1, r_2, angles θ_1, θ_2 and branch cuts in the definition of the complex potential (2.59).

Alternatively, we could insist that $x = 0$ be a streamline for the flow, that is

$$\operatorname{Im} w(z) = \psi = \text{constant} \quad \text{at } x = 0. \tag{2.55}$$

The two boundary conditions (2.54) and (2.55) are equivalent by virtue of (2.22).

The idea of the *method of images* is to insert a fictitious "image" source at the point $(-d, 0)$ behind the wall. By symmetry we would expect the flows caused by the two sources to cancel each other out on the symmetry line $x = 0$, and thereby satisfy the condition of zero normal velocity there. Of course, we will have introduced a new singularity, but it is outside the physical region $x > 0$ in which we are interested.

By superimposing the real and image sources, we obtain the velocity potential

$$w = \frac{Q}{2\pi}\log(z-d) + \frac{Q}{2\pi}\log(z+d), \tag{2.56}$$

and the corresponding velocity components are given by

$$u - \mathrm{i}v = \frac{\mathrm{d}w}{\mathrm{d}z} = \frac{Qz}{\pi\left(z^2 - d^2\right)}. \tag{2.57}$$

In particular, on the x-axis we have $z = \mathrm{i}y$ and hence

$$u - \mathrm{i}v\big|_{x=0} = \frac{-Q\mathrm{i}y}{\pi\left(y^2 + d^2\right)}, \tag{2.58}$$

so the boundary condition (2.54) of zero normal velocity is indeed satisfied.

To extract the velocity potential and streamfunction from the velocity potential (2.56), we have to confront the fact that $w(z)$ is a multifunction. We can expand each log in (2.56) to obtain

$$w(z) = \phi + \mathrm{i}\psi = \frac{Q}{2\pi}\left(\log r_1 + \log r_2\right) + \frac{\mathrm{i}Q}{2\pi}\left(\theta_1 + \theta_2\right), \tag{2.59}$$

where the lengths r_1, r_2 and angles θ_1, θ_2 are defined as shown in Figure 2.6. There is a branch cut extending to infinity from each of the branch points $z = \pm d$, and it

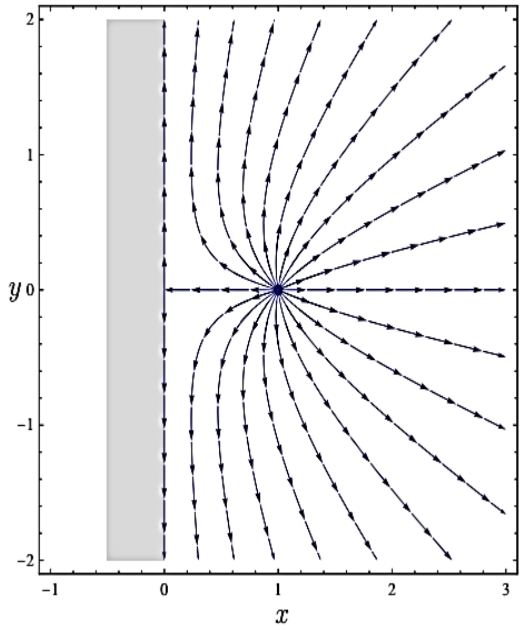

Figure 2.7: Streamlines for the flow produced by a source at the point $(1, 0)$ with an impermeable wall at $x = 0$.

is convenient to take them to lie along the x-axis as shown, so that $\theta_1 \in [0, 2\pi)$ and $\theta_2 \in (-\pi, \pi]$. We easily see that $\theta_1 + \theta_2 \equiv \pi$ on the y-axis, and hence the boundary condition (2.55) is satisfied by the complex potential (2.56).

We can read off from (2.59) the streamfunction for the flow and deduce that the streamlines are given by $\theta_1 + \theta_2 =$ constant. This may be rearranged to show that the streamlines are hyperbolae satisfying the equation

$$x^2 - 2\beta xy - y^2 = d^2, \tag{2.60}$$

where $\beta = \cot(2\pi\psi/Q)$ is constant. The resulting curves are plotted in Figure 2.7, with $d = 1$. Notice that the competition between the real and image sources causes a stagnation point at the origin.

This idea is easily extended to describe the flow due to a vortex in a half-space. The only difference is that, to achieve the necessary cancellation, the strength of the image vortex must be equal and opposite to that of the original vortex, as shown schematically in Figure 2.8. Thus, the complex potential due to a vortex of strength Γ at the point $(d, 0)$ in the half-space $x > 0$ with an impermeable boundary at $x = 0$ is

$$w(z) = -\frac{\mathrm{i}\Gamma}{2\pi}\log(z - d) + \frac{\mathrm{i}\Gamma}{2\pi}\log(z + d). \tag{2.61}$$

We can verify that this satisfies the required boundary condition at $x = 0$ by calculating the velocity field, using

$$u - \mathrm{i}v = \frac{\mathrm{d}w}{\mathrm{d}z} = \frac{-\mathrm{i}\Gamma d}{\pi\left(z^2 - d^2\right)}, \tag{2.62}$$

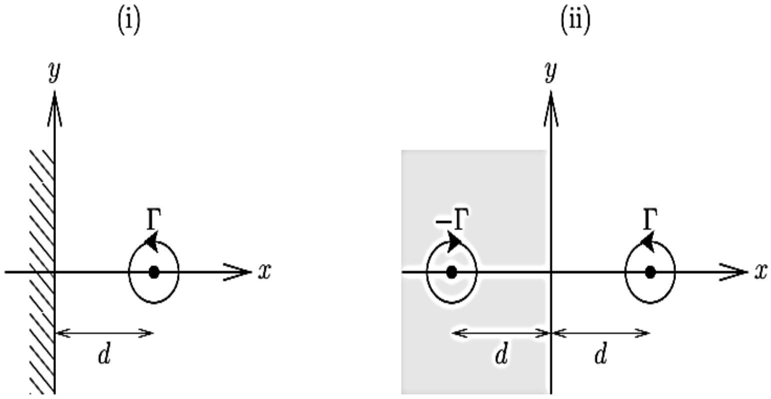

Figure 2.8: (i) Schematic of a vortex in the half-plane $x > 0$ bounded by a wall at $x = 0$. (ii) The same flow mimicked by inserting an image vortex.

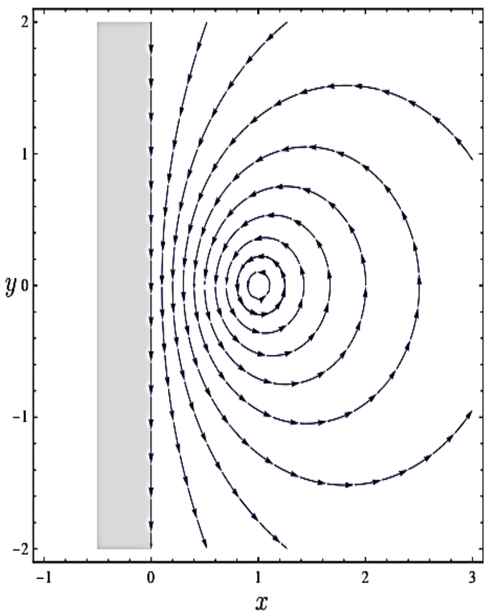

Figure 2.9: Streamlines for the flow produced by a vortex at the point $(1, 0)$ with an impermeable wall at $x = 0$.

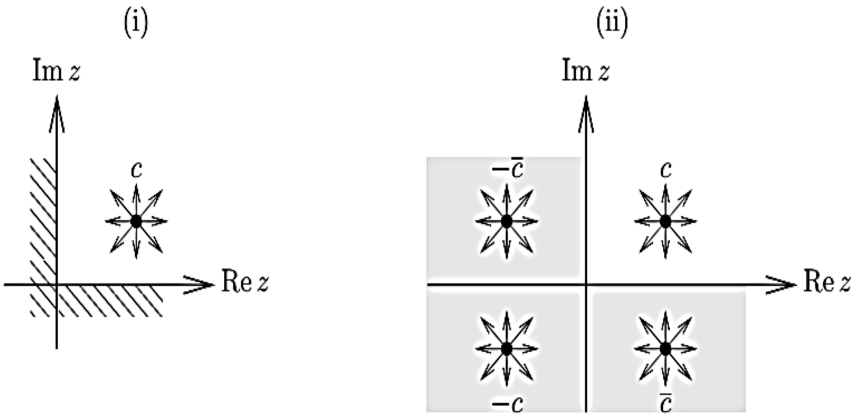

Figure 2.10: (i) Schematic of a source in the positive quadrant bounded by walls along the real and imaginary axes. (ii) The same flow mimicked by inserting three image sources.

which is clearly pure imaginary when $z = iy$.

The streamfunction for this flow is found from (2.61) to be given by

$$\psi = \frac{\Gamma}{2\pi} \log\left(\frac{r_2}{r_1}\right), \tag{2.63}$$

where r_1 and r_2 are again defined as in Figure 2.6. Hence the streamlines are curves on which $r_2/r_1 = \text{constant}$, and it is easily shown that these are circles centred on the x-axis, as shown in Figure 2.9.

The same ideas are easily extended to describe flow in a quadrant. In Figure 2.10(i), we show a source at the point representing the complex number $c = a + ib$ (with $a, b > 0$) in the Argand diagram, with impermeable boundaries along the real and imaginary axes. The condition of zero flow across the real axis may be satisfied by inserting an image source at $z = a - ib = \bar{c}$, where a bar is used to denote the complex conjugate. Now, to impose the equivalent boundary condition on the imaginary axis, we need to insert an image corresponding to each of the sources now present in $\operatorname{Re} z > 0$, at $z = -a + ib = -\bar{c}$ and at $z = -a - ib = -c$ respectively. We therefore end up with three image sources altogther, as shown in Figure 2.10(ii), and the resulting flow, with the required symmetry in both the real and imaginary axes, is represented by the complex potential

$$w = \frac{Q}{2\pi}\log(z-c) + \frac{Q}{2\pi}\log(z+c) + \frac{Q}{2\pi}\log(z-\bar{c}) + \frac{Q}{2\pi}\log(z+\bar{c}). \tag{2.64}$$

An analogous approach works for a vortex in a quadrant, provided the signs of the images are adjusted as shown in Figure 2.11. The resulting complex potential is

$$w = -\frac{i\Gamma}{2\pi}\log(z-c) - \frac{i\Gamma}{2\pi}\log(z+c) + \frac{i\Gamma}{2\pi}\log(z-\bar{c}) + \frac{i\Gamma}{2\pi}\log(z+\bar{c}). \tag{2.65}$$

In principle, the method of images may be extended to several other domains with straight boundaries. However, an infinite number of images will often be required to achieve symmetry in every boundary.

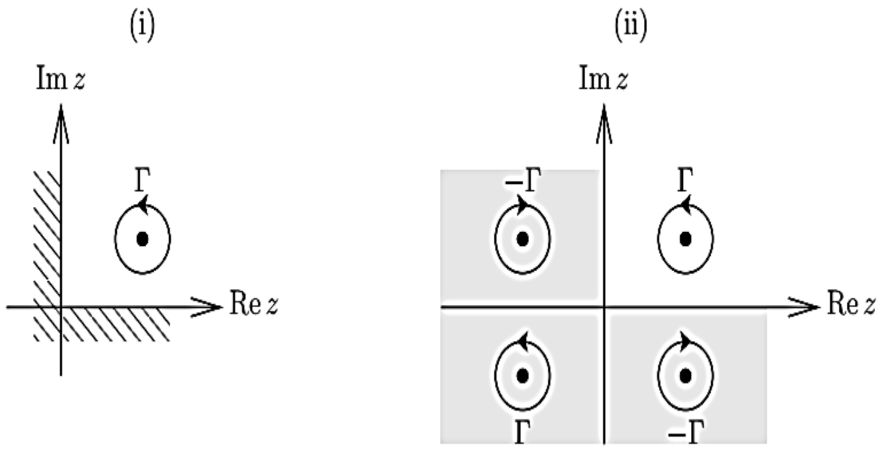

Figure 2.11: (i) Schematic of a vortex in the positive quadrant bounded by walls along the real and imaginary axes. (ii) The same flow mimicked by inserting three image vortices.

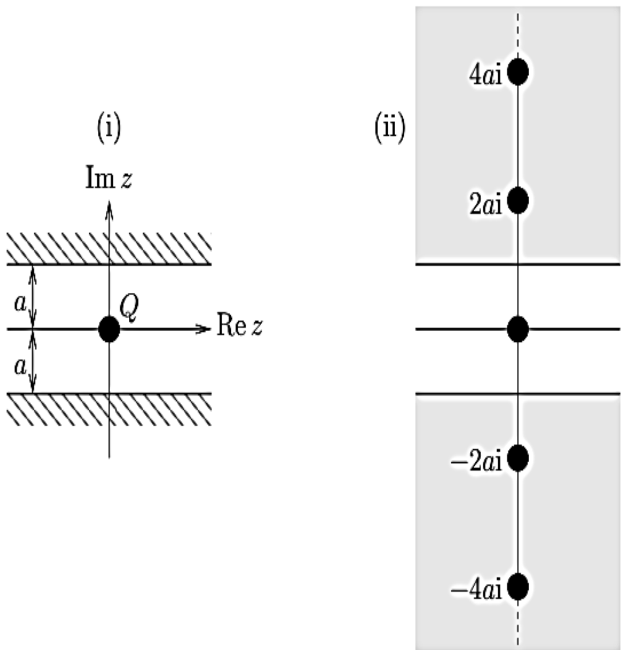

Figure 2.12: (i) Schematic of a source in the strip bounded by walls along the lines $\operatorname{Im} z = \pm a$. (ii) The same flow mimicked by inserting an infinite system of image sources.

Example 2.14 ***A source in a strip***

Fluid occupies the region $-a < y < a$ *between two rigid boundaries at* $y = \pm a$. *We will use the method of images to find the flow caused by a source placed in the fluid at the origin* $(x, y) = (0, 0)$, *as shown in* Figure 2.11(i).

At first glance, we can take care of the walls by inserting image sources at $z = \pm 2a\mathrm{i}$. *However, then we realise that the image at (say)* $z = 2a\mathrm{i}$ *needs its own image in the wall* $\operatorname{Im} z = -a\mathrm{i}$, *and we are therefore forced to include another image at* $z = -4a\mathrm{i}$. *By continuing this argument, we find that an infinite system of image sources along the imaginary axis is needed, at the points* $z = 2na\mathrm{i}$ *for all integers* n, *as shown in* Figure 2.11(ii).

The resulting velocity potential is apparently

$$w(z) = \frac{Q}{2\pi} \lim_{N\to\infty} \sum_{n=-N}^{N} \log(z - 2na\mathrm{i}), \tag{2.66}$$

although it is far from clear that this series converges! However, we can use it to calculate the velocity components, namely

$$u - \mathrm{i}v = \frac{\mathrm{d}w}{\mathrm{d}z} = \frac{Q}{2\pi}\left\{\frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{1}{z^2 + 4n^2a^2}\right\}. \tag{2.67}$$

This series does *converge, and may be evaluated using the identity*

$$\sum_{n=1}^{\infty}\frac{1}{n^2 + b^2} \equiv -\frac{1}{2b^2} + \frac{\pi\coth(b\pi)}{2b}. \tag{2.68}$$

We therefore find that

$$u - \mathrm{i}v = \frac{Q}{4a}\coth\left(\frac{\pi z}{2a}\right). \tag{2.69}$$

On the upper wall, where $z = \mathrm{i}a + x$, *we therefore find that*

$$u - \mathrm{i}v = \frac{Q}{4a}\tanh\left(\frac{\pi x}{2a}\right), \tag{2.70}$$

so the boundary condition $v = 0$ *on* $y = a$ *is indeed satisfied.*

Example 2.14 illustrates that in more complicated domains, although the method of images may work in principle, it is often unweildy in practice. As we will see, a far neater approach in such cases is to use *conformal mapping*. However, another example where the image system may be found relatively easily is where the boundary is a *circle*.