# LECTURE 3: POTENTIAL FLOW

## 1.5.1 Irrotational flow

A flow is said to be *irrotational* if the vorticity is identically zero:

$$\text{flow is irrotational} \quad \Leftrightarrow \quad \boldsymbol{\omega} \equiv \boldsymbol{\nabla} \times \boldsymbol{u} \equiv \boldsymbol{0}. \tag{1.53}$$

At first glance, this might seem like a far-fetched assumption. However, we note that the trivial solution $\boldsymbol{\omega} \equiv \boldsymbol{0}$ is consistent with the vorticity equation (1.46). Furthermore, we have just argued from Kelvin's Circulation Theorem that an initially irrotational flow must remain irrotational for all time. Therefore, it is difficult to create vorticity in an inviscid fluid, and actually rather likely that a flow will be irrotational.

For an irrotational flow, the momentum equation (1.39) simplifies to

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + \chi\right). \tag{1.54}$$

If the flow is steady, so the left-hand side is zero, we see that the bracketed term on the right-hand side must be constant and therefore deduce that

$$\frac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + \chi \text{ is constant } \textit{everywhere} \text{ in steady irrotational flow.} \tag{1.55}$$

This is **Bernoulli's Theorem for steady irrotational flow**, and should be compared with the weaker version (1.41) that holds for general steady flow.

## 1.5.2 The velocity potential

The Euler equations (1.36) and (1.37) are very difficult to solve in general, largely because of the nonlinear term $(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$ in the momentum equation. (In fact, the question of whether solutions of the Euler equations can develop singularities in finite time is hotly debated.) The problem becomes much more straightforward if the flow is irrotational, and we will assume henceforth that this is the case.

If $\boldsymbol{\nabla} \times \boldsymbol{u} \equiv \boldsymbol{0}$, there must exist a *velocity potential* $\phi(\boldsymbol{x}, t)$ such that

$$\boldsymbol{u} \equiv \boldsymbol{\nabla}\phi. \tag{1.56}$$

To prove this, we define

$$\phi(\boldsymbol{x}, t) := \phi_0(t) + \int_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}, \tag{1.57}$$

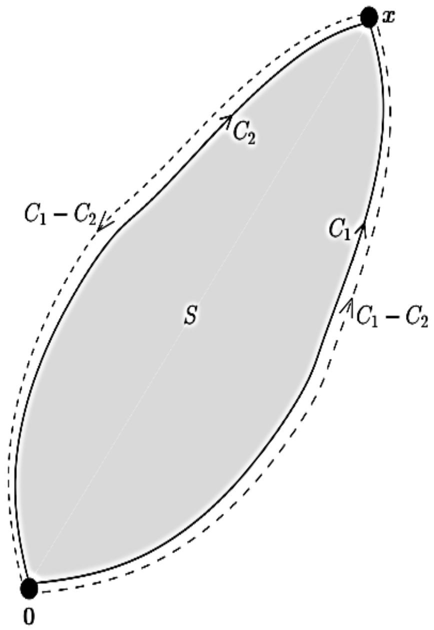


Figure 1.8: Schematic of two paths $C_1$ and $C_2$ joining the origin $\mathbf{0}$ to a point $\boldsymbol{x}$, along with the closed path $C_1 - C_2$ formed by joining them, spanned by a surface $S$.

where $C$ is any curve joining the origin to the point $\boldsymbol{x}$.

Note that $\phi$ is *not* unique: we can choose the scalar function $\phi_0(t)$ arbitrarily and (1.56) will still be satisfied. However, we will now show that the definition (1.57) is independent of the choice of the curve $C$. Let us consider two alternative paths $C_1$ and $C_2$ joining the origin to $\boldsymbol{x}$. Then

$$\int_{C_1} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} - \int_{C_1} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} \equiv \oint_{C_1 - C_2} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}, \tag{1.58}$$

where $C_1 - C_2$ is the closed circuit formed by joining $C_1$ and $C_2$ together, as illustrated in Figure 1.8. Now Stokes' Theorem gives

$$\oint_{C_1 - C_2} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} \equiv \iint_S (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \boldsymbol{n} \, \mathrm{d}S, \tag{1.59}$$

where $S$ is any surface spanning $C_1 - C_2$, and this is zero since the flow is assumed to be irrotational.

Hence $\phi$ is well defined by (1.57), up to the arbitrary function $\phi_0(t)$, and it is a simple exercise to show that $\phi$ then satisfies (1.56). Then the incompressibility condition (1.36) gives us

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} \equiv \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \phi) \equiv \nabla^2 \phi = 0, \tag{1.60}$$

so that $\phi$ satisfies *Laplace's equation*. This is very much easier to solve than the nonlinear Euler equations: given suitable boundary conditions, all the standard techniques such as separation of variables, transforms, *etc.* can be used to solve for $\phi$ and hence the velocity field $\boldsymbol{u}$.

The pressure may be found *a posteriori* from the momentum equation (1.35). This final step may be simplified as follows. With $\nabla \times \boldsymbol{u} \equiv 0$ and $\boldsymbol{u} \equiv \nabla\phi$, equation (1.39) becomes

$$\frac{\partial \nabla \phi}{\partial t} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}|\nabla\phi|^2 + \chi\right). \tag{1.61}$$

Since the $t$-derivative commutes with $\nabla$, we can rearrange this to

$$\nabla\left\{\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + \chi\right\} = \mathbf{0}. \tag{1.62}$$

It follows that the quantity in braces can be a function only of $t$, that is

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}|\nabla\phi|^2 + \chi = F(t) \text{ in irrotational flow.} \tag{1.63}$$

This generalisation of (1.55) is yet another version of Bernoulli's Theorem, namely **Bernoulli's Theorem for irrotational flow.**

Now we recall that the velocity potential is only defined up to an arbitrary function of $t$; if we define

$$\tilde{\phi} = \phi + f(t), \tag{1.64}$$

then $\tilde{\phi}$ is a potential corresponding to exactly the same velocity field through (1.56). In terms of $\tilde{\phi}$, Bernoulli's equation (1.63) becomes

$$\frac{\partial \tilde{\phi}}{\partial t} + \frac{p}{\rho} + \frac{1}{2}|\nabla\tilde{\phi}|^2 + \chi = F(t) - f'(t). \tag{1.65}$$

Hence the function $F(t)$ may be chosen arbitrarily by simply absorbing a suitable function of $t$ into $\phi$. For example, we can obtain (1.63) with $F(t) \equiv 0$ by choosing $f'(t) = F(t)$.

## 1.6 Background material

Here we list some of the standard results from Mods with which you should be familiar before starting this section of the course.

### 1.6.1 Vector calculus

Here we use the notation $\phi((x))$ to represent any differentiable scalar function, $\boldsymbol{u}(\boldsymbol{x})$ and $\boldsymbol{v}(\boldsymbol{x})$ any differentiable vector functions of the position vector $\boldsymbol{x} = (x, y, z)$. It is also sometimes handy to introduce suffix notation, so that $\boldsymbol{x}$ may be denoted by

$$\boldsymbol{x} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = x_1 e_1 + x_2 e_2 + x_3 e_3 = \sum_k x_k \boldsymbol{e}_k, \tag{1.66}$$

where we define $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$, $\boldsymbol{e}_1 \equiv \boldsymbol{i}$, $\boldsymbol{e}_2 \equiv \boldsymbol{j}$, $\boldsymbol{e}_3 \equiv \boldsymbol{k}$.

First we recall that the **grad** of a scalar function $\phi$ is defined by

$$\operatorname{grad} \phi \equiv \boldsymbol{\nabla}\phi := \boldsymbol{i}\frac{\partial \phi}{\partial x} + \boldsymbol{j}\frac{\partial \phi}{\partial y} + \boldsymbol{k}\frac{\partial \phi}{\partial z} \equiv \sum_k \boldsymbol{e}_k \frac{\partial \phi}{\partial x_k}; \tag{1.67}$$

the **divergence** and **curl** of a vector field $\boldsymbol{u}$ are defined by

$$\operatorname{div} \boldsymbol{u} \equiv \boldsymbol{\nabla} \cdot \boldsymbol{u} := \frac{\partial(\boldsymbol{i} \cdot \boldsymbol{u})}{\partial x} + \frac{\partial(\boldsymbol{j} \cdot \boldsymbol{u})}{\partial y} + \frac{\partial(\boldsymbol{k} \cdot \boldsymbol{u})}{\partial z} \equiv \sum_k \boldsymbol{e}_k \cdot \frac{\partial \boldsymbol{u}}{\partial x_k}; \tag{1.68}$$

$$\operatorname{curl} \boldsymbol{u} \equiv \boldsymbol{\nabla} \times \boldsymbol{u} := \frac{\partial(\boldsymbol{i} \times \boldsymbol{u})}{\partial x} + \frac{\partial(\boldsymbol{j} \times \boldsymbol{u})}{\partial y} + \frac{\partial(\boldsymbol{k} \times \boldsymbol{u})}{\partial z} \equiv \sum_k \boldsymbol{e}_k \times \frac{\partial \boldsymbol{u}}{\partial x_k}. \tag{1.69}$$

**Curl grad and div curl are zero**

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi) \equiv \boldsymbol{0}, \qquad \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) \equiv 0. \tag{1.70}$$

**Orthogonality of grad to level surfaces**

$$\boldsymbol{\nabla}\phi \text{ is normal to the surface is given by the equation } \phi(x, y, z) = \text{constant}. \tag{1.71}$$

**Directional derivative** The directional derivative of a scalar function $\phi(\boldsymbol{x})$ along the vector $\boldsymbol{u}$ is given by

$$\boldsymbol{u} \cdot (\boldsymbol{\nabla}\phi) \equiv (\boldsymbol{u} \cdot \boldsymbol{\nabla})\phi, \tag{1.72}$$

where $(\boldsymbol{u} \cdot \boldsymbol{\nabla})$ denotes the differential operator

$$(\boldsymbol{u} \cdot \boldsymbol{\nabla}) := (\boldsymbol{i} \cdot \boldsymbol{u})\frac{\partial}{\partial x} + (\boldsymbol{j} \cdot \boldsymbol{u})\frac{\partial}{\partial y} + (\boldsymbol{k} \cdot \boldsymbol{u})\frac{\partial}{\partial z} \equiv \sum_k (\boldsymbol{e}_k \cdot \boldsymbol{u})\frac{\partial}{\partial x_k}. \tag{1.73}$$

In this way, we can make sense of the directional derivative of a vector field $\boldsymbol{v}(\boldsymbol{x})$ without addressing the problem of defining the grad of a vector:

$$(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{v} := \sum_k (\boldsymbol{e}_k \cdot \boldsymbol{u})\frac{\partial \boldsymbol{v}}{\partial x_k}. \tag{1.74}$$

**Vector forms of the product rule**

$$\boldsymbol{\nabla} \cdot (\phi \boldsymbol{u}) \equiv (\boldsymbol{\nabla}\phi) \cdot \boldsymbol{u} + \phi(\boldsymbol{\nabla} \cdot \boldsymbol{u}), \tag{1.75a}$$

$$\boldsymbol{\nabla} \times (\phi \boldsymbol{u}) \equiv (\boldsymbol{\nabla}\phi) \times \boldsymbol{u} + \phi(\boldsymbol{\nabla} \times \boldsymbol{u}), \tag{1.75b}$$

$$\boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{v}) \equiv (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{v}) + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{u}), \tag{1.75c}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{v}) \equiv (\boldsymbol{\nabla} \cdot \boldsymbol{v})\boldsymbol{u} - (\boldsymbol{\nabla} \cdot \boldsymbol{u})\boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{v}. \tag{1.75d}$$

### 1.6.2 One-dimensional integrals

**Differentiation under the integral** Given a function $f(x,t)$, the integral

$$I(t) = \int_a^b f(x,t)\,\mathrm{d}x$$

is a function of $t$ alone. When computing the derivative of $I(t)$, the order of integration and differentation may be reversed, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b f(x,t)\,\mathrm{d}x \equiv \int_a^b \frac{\partial f}{\partial t}(x,t)\,\mathrm{d}x. \tag{1.76}$$

Note that the partial derivative $\partial f/\partial t$ is performed while *holding the integration variable $x$ constant.*

Equation (1.76) is a special case of **Leibnitz' rule**: if the limits $a$ and $b$ also depend on $t$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x \equiv \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x,t)\,\mathrm{d}x + f\big(b(t),t\big)\dot{b}(t) - f\big(a(t),t\big)\dot{a}(t). \tag{1.77}$$

### 1.6.3 Multi-dimensional integrals

**Line integrals** A space curve $C$ may be described using a single parameter, $\xi$ say, by $\boldsymbol{x} = \boldsymbol{x}(\xi)$ (i.e. $x = x(\xi)$, $y = y(\xi)$, $z = z(\xi)$), where $\xi$ lies in some interval, say $[a,b]$. Then, given a scalar field $\phi(\boldsymbol{x})$ and vector field $\boldsymbol{u}(\boldsymbol{x})$, we define

$$\int_C \phi\,\mathrm{d}\boldsymbol{x} := \int_a^b \phi\big(\boldsymbol{x}(\xi)\big)\frac{\mathrm{d}\boldsymbol{x}(\xi)}{\mathrm{d}\xi}\,\mathrm{d}\xi, \tag{1.78a}$$

$$\int_C \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{x} := \int_a^b \boldsymbol{u}\big(\boldsymbol{x}(\xi)\big)\cdot\frac{\mathrm{d}\boldsymbol{x}(\xi)}{\mathrm{d}\xi}\,\mathrm{d}\xi, \tag{1.78b}$$

$$\int_C \phi\,\mathrm{d}s := \int_a^b \phi\big(\boldsymbol{x}(\xi)\big)\left|\frac{\mathrm{d}\boldsymbol{x}(\xi)}{\mathrm{d}\xi}\right|\mathrm{d}\xi. \tag{1.78c}$$

The notation $\oint$ is sometimes used to distinguish integrals around closed curves, i.e. those where $\boldsymbol{x}(a) = \boldsymbol{x}(b)$.

**Surface integrals** A surface $S$ in $\mathbb{R}^3$ may be described using two parameters, say $\boldsymbol{x} = \boldsymbol{x}(\xi,\eta)$, where $(\xi,\eta)$ occupies some region $R \subseteq \mathbb{R}^2$. Then, given a scalar field $\phi(\boldsymbol{x})$

and vector field $\boldsymbol{u}(\boldsymbol{x})$, we define

$$\iint_S \phi \, \mathrm{d}S := \iint_R \phi(\boldsymbol{x}(\xi,\eta)) \left| \frac{\partial \boldsymbol{x}}{\partial \xi}(\xi,\eta) \times \frac{\partial \boldsymbol{x}}{\partial \eta}(\xi,\eta) \right| \mathrm{d}\xi \mathrm{d}\eta, \qquad (1.79a)$$

$$\iint_S \phi \, \mathrm{d}\boldsymbol{S} \equiv \iint_S \phi \boldsymbol{n} \, \mathrm{d}S := \iint_R \phi(\boldsymbol{x}(\xi,\eta)) \left( \frac{\partial \boldsymbol{x}}{\partial \xi}(\xi,\eta) \times \frac{\partial \boldsymbol{x}}{\partial \eta}(\xi,\eta) \right) \mathrm{d}\xi \mathrm{d}\eta, \qquad (1.79b)$$

$$\iint_S \boldsymbol{u} \, \mathrm{d}S := \iint_R \boldsymbol{u}(\boldsymbol{x}(\xi,\eta)) \left| \frac{\partial \boldsymbol{x}}{\partial \xi}(\xi,\eta) \times \frac{\partial \boldsymbol{x}}{\partial \eta}(\xi,\eta) \right| \mathrm{d}\xi \mathrm{d}\eta, \qquad (1.79c)$$

$$\iint_S \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S} \equiv \iint_S \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S := \iint_R \boldsymbol{u}(\boldsymbol{x}(\xi,\eta)) \cdot \left( \frac{\partial \boldsymbol{x}}{\partial \xi}(\xi,\eta) \times \frac{\partial \boldsymbol{x}}{\partial \eta}(\xi,\eta) \right) \mathrm{d}\xi \mathrm{d}\eta, \qquad (1.79d)$$

where $\boldsymbol{n}$ denotes the unit normal to $S$.

**Volume integrals** A volume $V$ in $\mathbb{R}^3$ may be described using three parameters, say $\boldsymbol{x} = \boldsymbol{x}(\xi,\eta,\zeta)$, where $(\xi,\eta,\zeta)$ occupies some region $R \subseteq \mathbb{R}^3$. Then, given a scalar field $\phi(\boldsymbol{x})$ and vector field $\boldsymbol{u}(\boldsymbol{x})$, we have

$$\iiint_V \phi \, \mathrm{d}x\mathrm{d}y\mathrm{d}z \equiv \iiint_V \phi \, \mathrm{d}V \equiv \iiint_R \phi(\boldsymbol{x}(\xi,\eta,\zeta)) J(\xi,\eta,\zeta) \, \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta, \qquad (1.80a)$$

$$\iiint_V \boldsymbol{u} \, \mathrm{d}x\mathrm{d}y\mathrm{d}z \equiv \iiint_V \boldsymbol{u} \, \mathrm{d}V \equiv \iiint_R \boldsymbol{u}(\boldsymbol{x}(\xi,\eta,\zeta)) J(\xi,\eta,\zeta) \, \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta, \qquad (1.80b)$$

where $J$ is the *Jacobian* of the transformation from $(\xi,\eta,\zeta)$ to $(x,y,z)$, that is

$$J(\xi,\eta,\zeta) \equiv \frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)} := \begin{vmatrix} \partial x/\partial \xi & \partial x/\partial \eta & \partial x/\partial \zeta \\ \partial y/\partial \xi & \partial y/\partial \eta & \partial y/\partial \zeta \\ \partial z/\partial \xi & \partial z/\partial \eta & \partial z/\partial \zeta \end{vmatrix}. \qquad (1.81)$$

### 1.6.4 Integral theorems

These are all multidimensional generalisations of the Fundamental Theorem of Calculus. For any volume $V$ with boundary $\partial V$ we have the **Divergence Theorem**

$$\iiint_V \boldsymbol{\nabla} \cdot \boldsymbol{u} \, \mathrm{d}x\mathrm{d}y\mathrm{d}z \equiv \iint_{\partial V} \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S, \qquad (1.82)$$

and a corollary of this is the identity

$$\iiint_V \boldsymbol{\nabla} \phi \, \mathrm{d}x\mathrm{d}y\mathrm{d}z \equiv \iint_{\partial V} \phi \boldsymbol{n} \, \mathrm{d}S. \qquad (1.83)$$

For any surface $S$ spanning a simple closed curve $C$, **Stokes' Theorem** states that

$$\iint_S (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \boldsymbol{n} \, \mathrm{d}S \equiv \oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}, \qquad (1.84)$$

where the orientation of the normal $\boldsymbol{n}$ is chosen such that $C$ rotates around it in a right-handed sense.