LECTURE 1: EQUATIONS OF MOTION

1.1 Introduction

In this section we will derive the equations of motion for an *inviscid* fluid, that is a fluid with zero viscosity. We begin by setting up the basic concepts that are needed to describe the motion of a continuous medium in three dimensions, and the fundamental *kinematic* equations relating the density to the deformation and the velocity. Next we derive the momentum equation for an inviscid fluid. By assuming that the fluid has constant density, we obtain a closed system of equations for the velocity and pressure, known as the *Euler equations.* We introduce the concepts of vorticity and circulation, and explain why it is reasonable to assume that most flows are *irrotational.* Finally, we will show that incompressible irrotational flow is governed by Laplace's equation.

It is worth emphasising that this course relies on familiarity with various concepts from Mods, in particular vector calculus and manipulations of multidimensional integrals. The basic material relevant to this section is collected in §1.6.

1.2 Kinematics

1.2.1 Preliminaries

The term *kinematics* refers to "the science of pure motion, considered without reference to the matter or objects moved or the force producing or changing the motion."[1] In this section, we will examine what can be said about the motion of *any* continuous medium, although we will often use the word "fluid" to help fix ideas. We will later restrict our attention to inviscid fluids in §1.3.

In a continuous medium, all state variables, such as density, velocity and pressure, are assumed to be continuous functions of position $\boldsymbol{x}$ and time t; in fact in this course we will assume that all dependent variables are continuously differentiable.

1.2.2 Eulerian and Lagragian variables

We can describe the motion of a fluid by tracking the position of each material "particle" or "element" as the medium deforms. Suppose that the fluid starts in some *reference state* at time zero before being subsequently deformed, so that a material point initially at position $\boldsymbol{X} = (X, Y, Z)$ is displaced to a new position $\boldsymbol{x} = (x, y, z)$ at time t, as illustrated in Figure 1.1. The initial position vector $\boldsymbol{X}$ defines the *Lagrangian* coordinates

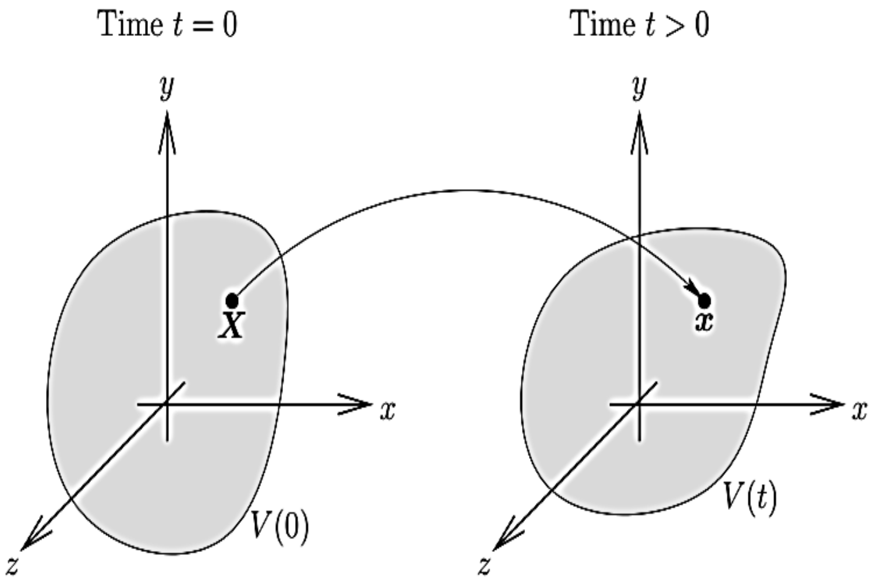

Figure 1.1: Schematic of the deformation of a fluid occupying a volume $V(t)$. The highlighted particle has *Eulerian* coordinate $\boldsymbol{x}$ and *Lagrangian* coordinate $\boldsymbol{X}$.

of each material element, while the current position vector $\boldsymbol{x}$ gives its *Eulerian* coordinates. A deformation of the medium corresponds to a mapping from each element's initial position to its current position at time t, that is a vector-valued transformation $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$. Our assumption that the medium is continuous implies that this mapping should be continuous and one-to-one, so that each element in the reference configuration is displaced continuously to a unique element in the deformed state. A sufficient condition for this to be true is[2]

$$0 < J < \infty, \tag{1.1}$$

where J is the *Jacobian* of the transformation, that is

$$J = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \frac{\partial(x,y,z)}{\partial(X,Y,Z)} = \begin{vmatrix} \partial x/\partial X & \partial x/\partial Y & \partial x/\partial Z \\ \partial y/\partial X & \partial y/\partial Y & \partial y/\partial Z \\ \partial z/\partial X & \partial z/\partial Y & \partial z/\partial Z \end{vmatrix}. \tag{1.2}$$

1.2.3 The convective derivative

In fluid dynamics, it is usually more convenient to use Eulerian variables, so we would write any property of the fluid (for example density, pressure, temperature, *etc.*) as a function of current position vector $\boldsymbol{x}$ and time t, say $f(\boldsymbol{x}, t)$. Then fixing $\boldsymbol{x}$ and letting t increase corresponds to following the time variation of f at a fixed point in space. Alternatively, we could write the same property as a function of Lagrangian variables by defining

$$F(\boldsymbol{X}, t) \equiv f\big(\boldsymbol{x}(\boldsymbol{X}, t), t\big). \tag{1.3}$$

Now, fixing $\boldsymbol{X}$ corresponds to tracking how f varies for a particular material element that moves with the deforming fluid.

These two viewpoints prompt us to define two different time derivatives. We use the usual partial derivative notation to denote the Eulerian time derivative, at a fixed position $\boldsymbol{x}$ in space, that is

$$\frac{\partial}{\partial t} := \left.\frac{\partial}{\partial t}\right|_{\boldsymbol{x}} = \text{rate of change with } \boldsymbol{x} \text{ held constant.} \tag{1.4}$$

On the other hand, we introduce the notation

$$\frac{\mathrm{D}}{\mathrm{D}t} := \left.\frac{\partial}{\partial t}\right|_{\boldsymbol{X}} = \text{rate of change with } \boldsymbol{X} \text{ held constant,} \tag{1.5}$$

to denote the Lagrangian time derivative (the notation $\mathrm{d}/\mathrm{d}t$ is also often employed). With $\boldsymbol{X}$ held constant, $\mathrm{D}/\mathrm{D}t$ corresponds to the rate of change following an element that convects with the fluid, and it is referred to as the *convective derivative* or the *material derivative* or sometimes the *derivative following the flow.*

1.2.4 Velocity and acceleration

The *velocity* $\boldsymbol{u}$ of the fluid is simply the rate of change of the position vector $\boldsymbol{x}$ for a material element, that is

$$\boldsymbol{u} = \frac{\mathrm{D}\boldsymbol{x}}{\mathrm{D}t}. \tag{1.6}$$

In general, the velocity will be a function of both position and time, so that $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$. If $\boldsymbol{u}$ does *not* depend on t, then the we say that the flow is *steady.*

We can now relate the Eulerian and Lagrangian time derivatives in terms of the velocity $\boldsymbol{u}$. The chain rule implies that

$$\frac{\mathrm{D}f}{\mathrm{D}t}(\boldsymbol{x},t) = \frac{\mathrm{D}t}{\mathrm{D}t}\frac{\partial f}{\partial t}(\boldsymbol{x},t) + \frac{\mathrm{D}\boldsymbol{x}}{\mathrm{D}t}\cdot\boldsymbol{\nabla} f(\boldsymbol{x},t) \tag{1.7}$$

and hence

$$\frac{\mathrm{D}f}{\mathrm{D}t} \equiv \frac{\partial f}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} f, \tag{1.8}$$

for any differentiable function f. We can write this in operator form as

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla}), \tag{1.9}$$

where $(\boldsymbol{u}\cdot\boldsymbol{\nabla})$ is shorthand for the directional derivative

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla}) \equiv u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{1.10}$$

and $\boldsymbol{u} = (u, v, w)$ are the components of the velocity.

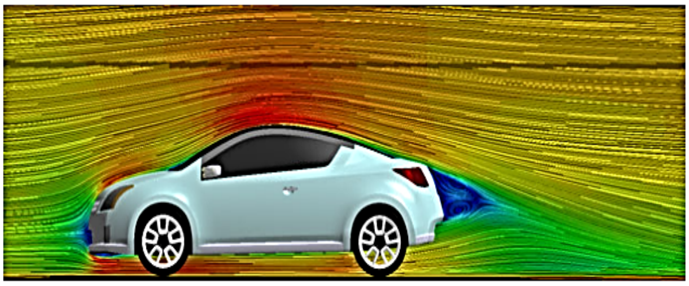

Figure 1.2: Streamlines for air flow over a car.

In particular, we can now compute the *acceleration* of the fluid, namely the material rate of change of the velocity:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \equiv \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}. \tag{1.11}$$

Note the way that the second term is grouped: $(\boldsymbol{u}\cdot\boldsymbol{\nabla})$ is a linear scalar differential operator which can easily be applied to the vector $\boldsymbol{u}$. Had we instead written this term as $\boldsymbol{u}\cdot(\boldsymbol{\nabla}\boldsymbol{u})$ we would have faced the problem of defining the grad of a vector. This can be done, but is to be avoided throughout this course.

1.2.5 Flow visualisation

If we know the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$, there are several ways of trying to visualise it. One is to plot the *streamlines*. This corresponds to taking a snapshot of the flow at a fixed time t, then plotting curves that are everywhere parallel to the velocity field $\boldsymbol{u}$. This results in the sort of plots that we often see in car commercials, for example: see Figure 1.2.

With t held constant, we can construct a curve that is everywhere parallel to $\boldsymbol{u}$ by solving the simultaneous differential equations

$$\frac{\mathrm{d}\boldsymbol{x}(s)}{\mathrm{d}s} = \boldsymbol{u}\big(\boldsymbol{x}(s),t\big).$$

By solving these differential equations with different initial conditions, we will obtain a family of streamlines.

Example 1.1 *The two-dimensional velocity field* $\boldsymbol{u} = (x,-y,0)$ *is called a* stagnation point flow. *We find the streamlines by solving the differental equations*

$$\frac{\mathrm{d}x}{\mathrm{d}s} = x, \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = -y, \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = 0.$$

The solution is easily found to be

$$x = A\mathrm{e}^{s}, \qquad y = B\mathrm{e}^{-s}, \qquad z = C,$$

where A, B *and* C *are integration constants. These parametrise the hyperbolae* $xy = AB = \text{const}$ *in the* (x,y)*-plane, as illustrated in* Figure 1.3.

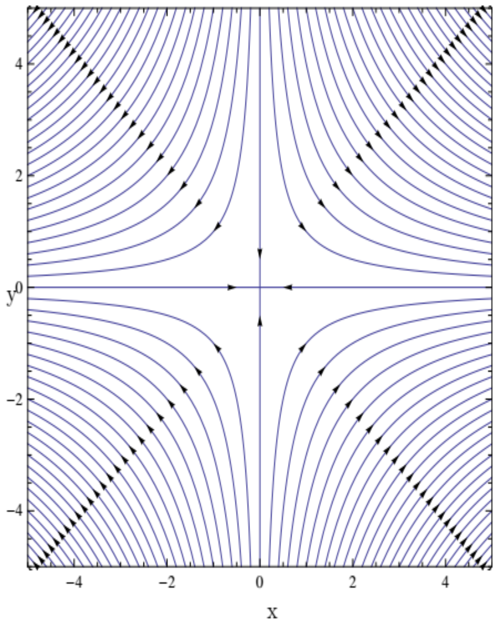

Figure 1.3: Streamlines for a stagnation point flow.

Plotting streamlines is similar to plotting phase plane trajectories for plane autonomous differential equations. The streamlines have a unique tangent vector equal to $\boldsymbol{u}$ at each point in the flow. Hence they can only cross at so-called *stagnation points*, where $\boldsymbol{u} = \mathbf{0}$ and the fluid is locally stationary. In Example 1.1, there is just one stagnation point at the origin, and it resembles a saddle point in a phase plane.

An alternative flow visualisation strategy, often used in experiments, is to insert tiny tracer particles into the flow and follow their trajectories. Assuming that each particle moves with the local flow velocity, its position vector $\boldsymbol{x}(t)$ must satisfy the differential equation

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{u}\big(\boldsymbol{x}(t), t\big). \tag{1.12}$$

Solutions of this equation are called *particle paths* for the flow $\boldsymbol{u}(\boldsymbol{x}, t)$.

If the flow is steady (*i.e.* $\boldsymbol{u}$ is independent of t), then we see that the streamline pattern will not vary with time, and that the streamlines and particle paths will coincide. For an unsteady flow, though, the streamline pattern will in general vary with time and not coincide with the particle paths.

Example 1.2 *For the two-dimensional unsteady flow* $\boldsymbol{u} = (\cos t, \sin t, 0)$*, the streamlines satisfy*

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos t, \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = \sin t, \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = 0, \tag{1.13}$$

with t *held fixed, and hence are given by*

$$x = A + s\cos t, \qquad y = B + s\sin t, \qquad z = C,$$

where A*,* B *and* C *are integration constants. These give a family of parallel straight lines in the* (x, y)*-plane whose direction rotates as* t *varies, as shown in* Figure 1.4.

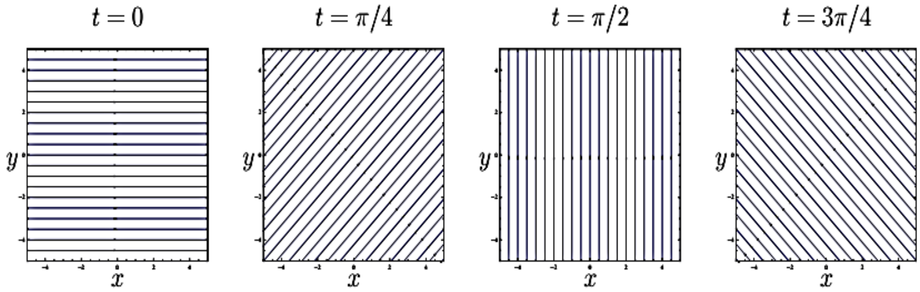

Figure 1.4: Streamlines for the unsteady flow (1.13).

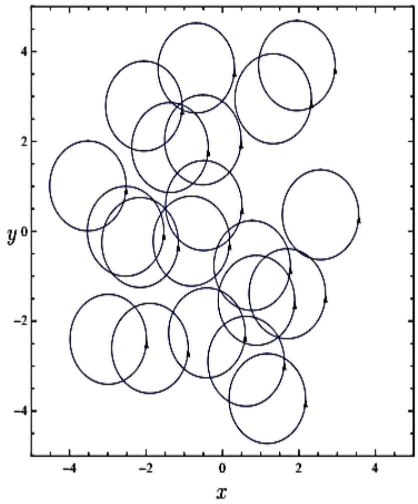

Figure 1.5: Particle paths for the unsteady flow (1.13).

The particle paths satisfy the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \cos t, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \sin t, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = 0, \tag{1.14}$$

and hence are given by

$$x = A + \sin t, \qquad y = B - \cos t, \qquad z = C,$$

for some constants A, B, C. Particles therefore trace out unit circles in the (x, y)-plane; some examples are shown in Figure 1.5.

1.2.6 Euler's identity

Now we derive an important result showing how the divergence of the velocity is related to expansion or contraction of the fluid. The derivation is lengthy and non-examinable but included here for completeness. We start by differentiating the definition (1.2) of J

with respect to t, differentiating each row of the determinant in term to obtain

$$\frac{\mathrm{D}J}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{vmatrix} = \begin{vmatrix} \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial x}{\partial X}\right) & \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial x}{\partial Y}\right) & \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial x}{\partial Z}\right) \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{vmatrix}$$

$$+ \begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial y}{\partial X}\right) & \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial y}{\partial Y}\right) & \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial y}{\partial Z}\right) \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{vmatrix} + \begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial z}{\partial X}\right) & \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial z}{\partial Y}\right) & \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial z}{\partial Z}\right) \end{vmatrix}. \tag{1.15}$$

For convenience we denote the three determinants on the right-hand side of (1.15) by Δ_1, Δ_2 and Δ_3 respectively. Since the convective derivative is taken with $\boldsymbol{X}$ fixed, it commutes with X-, Y- and Z-derivatives. Recalling also that $\mathrm{D}x/\mathrm{D}t = u$, we can rewrite Δ_1 as

$$\Delta_1 = \begin{vmatrix} \frac{\partial u}{\partial X} & \frac{\partial u}{\partial Y} & \frac{\partial u}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{vmatrix}. \tag{1.16}$$

We apply the chain rule to each of the derivatives in the first row to obtain

$$\Delta_1 = \frac{\partial u}{\partial x}\begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{vmatrix} + \frac{\partial u}{\partial y}\begin{vmatrix} \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{vmatrix} + \frac{\partial u}{\partial z}\begin{vmatrix} \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{vmatrix}. \tag{1.17}$$

The final two determinants in (1.17) have repeated rows and are therefore identically zero. It follows that

$$\Delta_1 = \frac{\partial u}{\partial x}J, \tag{1.18}$$

and analogous manipulations lead to

$$\Delta_2 = \frac{\partial v}{\partial y}J, \qquad \Delta_3 = \frac{\partial w}{\partial z}J. \tag{1.19}$$

By substituting these into (1.15), we obtain *Euler's identity*

$$\frac{\mathrm{D}J}{\mathrm{D}t} = J\boldsymbol{\nabla}\cdot\boldsymbol{u}. \tag{1.20}$$

Recall that the Jacobian relates infinitesimal volumes in the Eulerian and Lagrangian frames, via $\mathrm{d}x\mathrm{d}y\mathrm{d}z = J\mathrm{d}X\mathrm{d}Y\mathrm{d}Z$. Hence we can interpret J as measuring the local expansion or contraction: the fluid is expanding if J is increasing with time or contracting if J decreases with time. The identity (1.20) shows how this local expansion or contraction of the medium is related to the divergence of the velocity. A flow is said to be *incompressible* or *volume-preserving* if it preserves infinitesimal volumes, that is if $\mathrm{D}J/\mathrm{D}t \equiv 0$. From (1.20), we see that

$$\text{flow is incompressible} \quad \Leftrightarrow \quad \boldsymbol{\nabla} \cdot \boldsymbol{u} \equiv 0. \tag{1.21}$$