LECTURE 11:

BIFURCATIONS IN 2-D MAPS

We prelude our discussion of 2-D nonlinear maps with a discussion of linear 2-D maps.

3.1 Linear 2-D Maps

Linear maps on $\mathbb{R}^2$ are linear transformations defined by 2×2 matrices ie. $\underline{v} \mapsto A\underline{v}$, where A is a 2×2 matrix with

$$A(a\underline{v} + b\underline{w}) = aA\underline{v} + bA\underline{w},$$

for $a, b \in \mathbb{R}$, $\underline{v}, \underline{w} \in \mathbb{R}^2$ and

$$A\underline{v} = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Every linear map has a fixed point at the origin.

Proof. Exercise. □

Recall: λ is an eigenvalue of A if $\exists \underline{v} \neq 0$ such that $A\underline{v} = \lambda\underline{v}$. Suppose $\underline{v}_0$ is an eigenvector with eigenvalue λ. Then $A\underline{v}_0 = \lambda\underline{v}_0$.

Applying A times gives

$$\underline{v}_1 = A\underline{v}_0 = \lambda\underline{v}_0$$

$$\underline{v}_2 = A\underline{v}_1 = A\lambda\underline{v}_0 = \lambda A\underline{v}_0 = \lambda^2\underline{v}_0.$$

$\therefore$

$$\underline{v}_n = \lambda^n\underline{v}_0 \quad \text{and} \quad \underline{v}_{n+1} = A\underline{v}_n,$$

so that it behaves like a 1-D map under iteration.

Example Suppose $A \in \mathbb{R}^2$ is defined by

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}.$$

Then

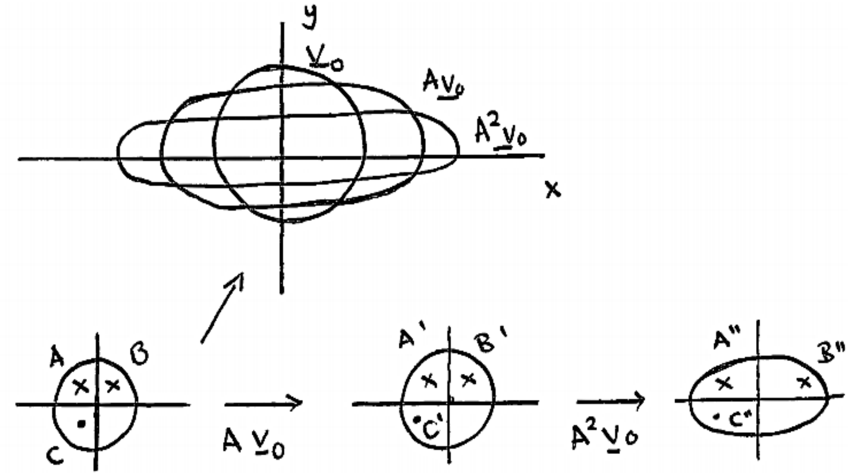
$$A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$$

so that

$$A^n\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} a^n x_0 \\ b^n y_0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

It follows that the unit disc is mapped into an ellipse with axes $|a|^n$ and $|b|^n$ along x and y respectively.

- If both $|a| < 1$ and $|b| < 1$, the axes shrink as $n \to \infty$, and we have a sink.
- If $|a| > 1 > |b|$, the x-axis ($|a|^n$) grows, while the y-axis ($|b|^n$) shrinks as $n \to \infty$, and we have a saddle.
- If $|a| > 1$ and $|b| > 1$, both axes expand as $n \to \infty$, and we have a source.



The determinant $\det A$ measures the area transformed by A.

Definition 6. *If* $\det A = \pm 1$, *we have an area-preserving map.*

Definition 7. *A is a hyperbolic map if no eigenvalues have magnitude one.*

Lemma 2. *If $A(\underline{v})$ is a linear map on $\mathbb{R}^2$, then*

1. *The origin is a sink if all eigenvalues of A have magnitude < 1.*

2. *The origin is a source if all eigenvalues of A have magnitude > 1.*

3. *If a hyperbolic map has eigenvalues > 1 and < 1, the origin is a saddle.*

3.2 Nonlinear Maps and Jacobians

Consider $\underline{x}_{n+1} = \underline{F}(\underline{x}_n)$ with fixed pt $\underline{X} = F(\underline{X})$.
Perturb away from $\underline{X}$: $\underline{\xi}_n = \underline{x}_n - \underline{X}$.
Then

$$\underline{x}_{n+1} = \underline{\xi}_{n+1} + \underline{X} = F(\underline{\xi}_n + \underline{X}) = F(\underline{X}) + D\underline{F}.\underline{\xi}_n + o(|\underline{\xi}_n|),$$

as $\underline{\xi}_n \to 0$, where

$$D\underline{F}(\underline{X}) = [\nabla \underline{F}]_{\underline{X}} = [\frac{\partial f_i}{\partial x_j}]\Big|_{\underline{X}} = J$$

is the Jacobian matrix of $\underline{F}$, evaluated at $\underline{X}$.
Thus

$$\underline{\xi}_{n+1} = J\underline{\xi}_n$$

and so stability reduces to determining the nature of fixed points of a linearised 2-D map, J. This result readily extends to n-D.

3.3 Periodic Orbits

Suppose A has a periodic orbit $\{\underline{X}_1, \ldots, \underline{X}_p\} = \underline{S}$ of period p. Then $\underline{X}_{r+1} = \underline{F}(\underline{X}_r)$ for $r = 1, \ldots, p-1, p$ with $\underline{X}_1 = \underline{F}(\underline{X}_p)$ where

$$\underline{F}(\underline{S}) = \{\underline{F}(\underline{X}_1), \ldots, \underline{F}(\underline{X}_p)\} = \{\underline{X}_2, \ldots, \underline{X}_1\}.$$

Thus $\underline{S}$ is an invariant set under $\underline{F}$.
To consider the stability of a p-cycle, we consider the p^{th} iteration of the map.

3.4 2-Cycles

Consider the 2-cycle $\{\underline{X}, \underline{Y}\}$ such that $\underline{Y} = \underline{F}(\underline{X})$, $\underline{X} = \underline{F}(\underline{Y})$. Then

$$\underline{G}(\underline{X}) = \underline{F}^2(\underline{X}) = \underline{F}(\underline{F}(\underline{X})).$$

A fixed point of $\underline{G}(\underline{X}) = \underline{X} = \underline{F}(\underline{F}(\underline{X}))$ is either a fixed point of $\underline{F}$ or a period-2 cycle.

Proof: Exercise
Suppose $\underline{F}$ is twice continuously differentiable. Then, linearising $\underline{G}$ about $\underline{X}$:

$$\underline{\xi}_{n+1} = \left[\frac{\partial G_i}{\partial x_j}\right]_{\underline{X}} \underline{\xi}_n = \underline{K}(\underline{X})\xi_n.$$

However stability is given by the chain rule:

$$\underline{K}(\underline{X}) = D\underline{G} = D(\underline{F}(\underline{F}(\underline{X}))) = D\underline{F}(\underline{F}(\underline{X}))D\underline{F}(\underline{X}) = D\underline{F}(\underline{Y})D\underline{F}(\underline{X}).$$

Therefore

$$\underline{K}(\underline{X}) = \underline{J}(\underline{F}(\underline{X}))\underline{J}(\underline{X}) = \underline{J}(\underline{Y})\underline{J}(\underline{X}).$$

Thus for 2-D maps, the stability of $\underline{X}$ and $\underline{Y}$ is given by the eigenvalues of the product of two Jacobians, $DF(\underline{X})$ and $DF(\underline{Y}$, evaluated at $\underline{X}$ and $\underline{Y}$.

If f is a 1-1 map, its inverse f^{-1} is a function and is defined.

3.5 Lyapunov Exponents and Numbers

Above we saw that perturbations $\xi_n = x_n - x_*$ from the fixed point x_* of a 1-D map satisfy, under iteration on n

$$\xi_n = |f'(x_*)|^n \xi_0.$$

Thus, when $f'(x_*) > 1$, the orbit of each point close to x_* diverges at a multiplicative rate of approximately $f'(x_*)$.

For a periodic-cycle of period K, stability is given by $f'(x_1)f'(x_2)\dots f'(x_K) = A$. If $A > 1$, the orbit or each neighbour x of the period K-cycle diverges at a rate of $\approx A$ after K iterations. The average multiplicative rate of separation is $A^{1/k}$ per iterate.

Definition 8. *The Lyapunov number (L.N.) quantifies this average rate of separation of its very close to* x_*.

Definition 9. *The Lyapunov exponent (L.E.) is the natural logarithm of the Lyapunov number.*

Examples

1. A L.N. of 2 (ie. L.E $\ln 2$) for the orbit of x_* means the distance between the orbit of x_* means the distance between the orbit of x_* and of $x_* + \xi$ for $|\xi| << 1$, will double after each iteration.
2. A L.N. of $1/2$, means the instance halves at each iteration and the orbits of x_* and $x_* + \xi$ move rapidly together.
3. A L.N. of 1 means no change: its LE$= 0$.

   L.N.s and L.E.s are important when applied to chaotic (non periodic) orbits, where there is sensitive dependence on i.c.

Definition 10. *Let f be a smooth map $\mathbb{R} \to \mathbb{R}$. The L.N. $L(x_1)$ of the orbit $\{x_1, x_2, \dots\}$ is defined as*

$$LN(x_1) = \lim_{n\to\infty} \left(|f'(x_1)||f'(x_2)|\dots|f'(x_n)|\right)^{1/n},$$

if this limit exists and

$$L.E.(x_1) = \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} \ln|f'(x_m)|.$$

Since $\ln L = L.E.$, the L.E. exists if LN exists and is non-zero.

N.B.

1. The L.N. of a fixed pt x_* for a 1-D map is $|f'(x_*)|$ and its $L.E.$ is $\ln f'(x_1)$.
2. The L.E. of a period K-cycle is

$$L.E.(x_1) = \frac{1}{K}\sum_{m=1}^{K} \ln|f'(x_m)|,$$

and the L.N., $L(x_1) = e^{L.E.(x_1)}$ describes the average local stretching per iterate.

3. A chaotic orbit has $L.E. > 0$.

Eg.: For the Bernoulli shift map $f(y) = 2ymod1$, $(y_{n+1} = 2y_n mod1)$, the $L.E.$ is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \ln |f'(y_m)| = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \ln 2 = \ln 2.$$