LECTURE 7:

RELAXATION OSCILLATORS AND TRANSITION LAYERS

We now consider the type of periodic solution that the Van der Pol equation possesses.

3.1 Relaxation Oscillations

We return to the Van der Pol Oscillator equation

$$\ddot{x} + \Lambda(x^2 - 1)\dot{x} + x = 0, \tag{3.1}$$

in the limit when we have a large parameter,$\Lambda >> 1$. Rescaling t with Λ, so that $t = \Lambda T$, we get

$$\delta x'' + (x^2 - 1)x' + x = 0, \tag{3.2}$$

where $\delta = \frac{1}{\Lambda^2} << 1$, $t = \Lambda T$ and $\frac{d}{dT} =\prime$.

Now

$$\delta x'' + x^2 x' - x' = \frac{d}{dT}[\delta x' + \frac{1}{3}x^3 - x].$$

We can therefore introduce

$$y = \delta x' + \frac{1}{3}x^3 - x$$

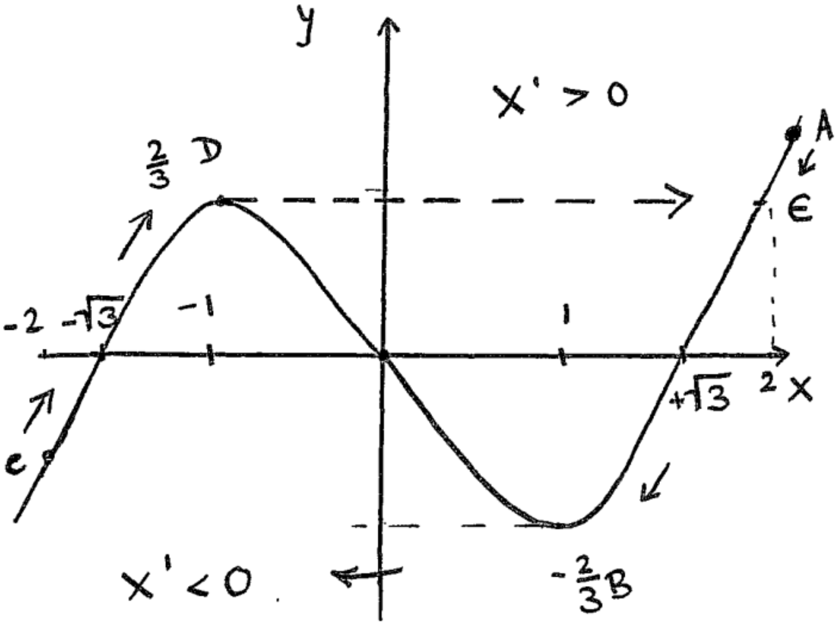
to get

$$\begin{aligned} y' &= -x, && (3.3a) \\ \delta x' &= y - (\frac{1}{3}x^3 - x). && (3.3b) \end{aligned}$$

Since $\delta << 1$, (7.3b) implies that

$$y \approx \frac{1}{3}x^3 - x = y_q, \tag{3.4}$$

ie. x rapidly evolves to a state of quasi-equilibrium, given by y_q:



From (7.4), $y' = x^2 - 1 = 0$ when $x = \pm 1$ and $y = \mp\frac{2}{3}$.
On y_q, (7.3a) gives $\dot{y} = -x_q(y)$.
At A, $x_q > 0 \Rightarrow \dot{y} < 0 \Rightarrow y$ decreases until $x_q = 1$ at B.
At B, there is a rapid transitional jump to C in a time $\sim \delta$.
At C, $x_q < 0 \Rightarrow \dot{y} > 0 \Rightarrow y$ increases until D, where $x_q = -1$.
At D there exists a rapid transitional jump to E.

At E, (7.4) implies that $\frac{1}{3}x^3 - x = \frac{2}{3}$ so that

$$x^3 - 3x - 2 = 0 = (x+1)^2(x-2) = 0.$$

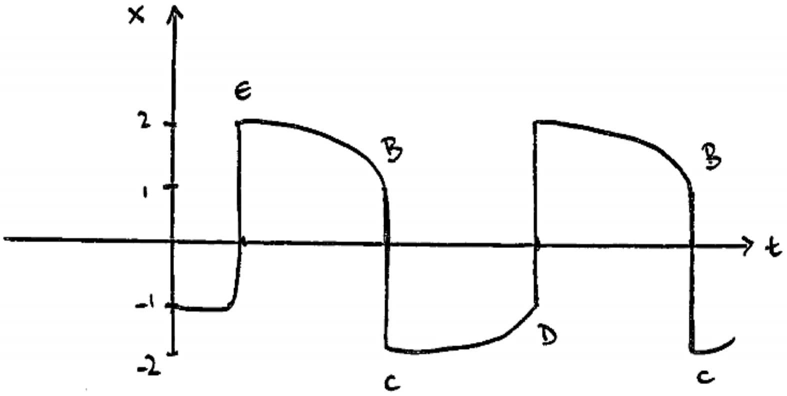
Thus $x = 2$ at E.

Similarly, at C, $\frac{1}{3}x^3 - x = -\frac{2}{3}$, so that

$$x^3 - 3x + 2 = 0 = (x-1)^2(x+2) = 0$$

and therefore $x = -2$.

This is called a relaxation oscillation, and is formed from the periodic cycle $EBCDE$. It has an associated time series, whose maximum amplitude varies between $+2$ at E and -2 at C:



To estimate the period of the oscillation, we can use the symmetry $\overrightarrow{EB} = \overrightarrow{CD}$. So the period is approximately twice the time taken between E and B:

$$P_{er} = 2\int_E^B dt.$$

Using $\frac{dy}{dt} = y'$, we obtain

$$P_{er} = 2\int_{2/3}^{-2/3} \frac{dy}{y'}.$$

Since $y' = -x_q$ and $dy = \frac{dy_q}{dx}dx$, we obtain

$$P_{er} = 2\int_2^1 -\frac{1}{x}\frac{dy_q}{dx}dx$$

and using $y' = x^2 - 1$,

$$P_{er} = 2\int_1^2 (x^2-1)\frac{dx}{x} = 2\left[\frac{1}{2}x^2 - \ln x\right]_1^2 = 3 - 2\ln 2.$$

Thus in the original time units:

$$\text{period } T \sim 2\pi \text{ for } \Lambda << 1 \Rightarrow \text{ period } t \sim (3 - 2\ln 2)\Lambda \text{ for } \Lambda >> 1. \qquad (3.5)$$

3.2 Transition Layers

Consider (7.2):

$$\delta x'' + (x^2 - 1)x' + x = 0,$$

in the slow part of the oscillation, i.e. EB or CD. In this region set $\delta = 0$ so that:

$$(x^2 - 1)x' + x = 0. \qquad (3.6)$$

We seek a regular perturbation series solution:

$$x = x_0 + \delta x_1 + \cdots .$$

At leading order we obtain

$$\mathcal{O}(1)(x_0^2-1)x_0' + x_0 = 0 = x_0x_0' - \frac{x_0}{x_0'} + 1 = 0,$$

which can be integrated to give:

$$\frac{1}{2}x_0^2 - \ln|x_0| = -t + \text{const.}$$

Suppose: $x_0 = \pm 1$ at $t = t^*$ (i.e. at the transition points B and D).
Then

$$\frac{1}{2} = -t^* + \text{const.} \Rightarrow \quad \text{const.} = t^* + \frac{1}{2}.$$

Therefore

$$\frac{1}{2}x_0^2 - \ln|x_0| = \frac{1}{2} - (t - t^*) \tag{3.7}$$

Near $t = t^*$, $x_0 \pm (1+\xi)$, $|\xi| << 1$, so that (7.7) becomes

$$\frac{1}{2} + \xi + \frac{1}{2}\xi^2 - \left(\xi - \frac{\xi^2}{2} + \cdots\right) = \frac{1}{2} - (t - t^*),$$

i.e.

$$\xi^2 \sim (t^* - t) \Rightarrow \xi \sim \pm[(t^* - t)]^{1/2},$$

provided $t^* > t$.
We need to match the solution in EB to the 2 rapid transition solutions BC and DE.

Near B: put

$$t = t_B + \delta s \Rightarrow dt = \delta ds \Rightarrow \delta\frac{d}{dt} = \frac{d}{ds}.$$

Then (7.2)×δ becomes

$$x_{ss} + (x^2-1)x_s + \delta x = 0. \tag{3.8}$$

Try $x = X_0 + \delta X_1 + \cdots$, so that

$$X_0'' + (X_0^2 - 1)X_0' = 0. \tag{3.9}$$

We need to match this solution to the above solution using the condition: $X_0 \to 1$, as $s \to -\infty$ (at B) and $X_0' \to 0$ (since we have a minimum.)

Integrating (7.9) wrt s gives:

$$X_0' + \frac{1}{3}X_0^3 - X_0 = c.$$

Using the matching conditions,

$$0 + \frac{1}{3} - 1 = c \Rightarrow c = -\frac{2}{3}.$$

Therefore

$$X_0' = -\frac{2}{3} - (\frac{1}{3}X_0^3 - X_0) = -\frac{1}{3}[(X_0-1)^2(X_0+2)]. \tag{3.10}$$

Now point C corresponds to $x = -2$ and point E to $x = 2$.
Thus $X_0 \to -2$ as $s \to \infty$, i.e $x = -2$ at $t = t_c$. Thus by (7.7) (with $t^* = t_B$, $t = t_C$),

$$t_B - t_C = -\frac{1}{2} + \frac{1}{2}x_0^2 - \ln|x_0|\Big|_{x=-2} = \frac{3}{2} - \ln 2,$$

and the period of the oscillation is again seen to be

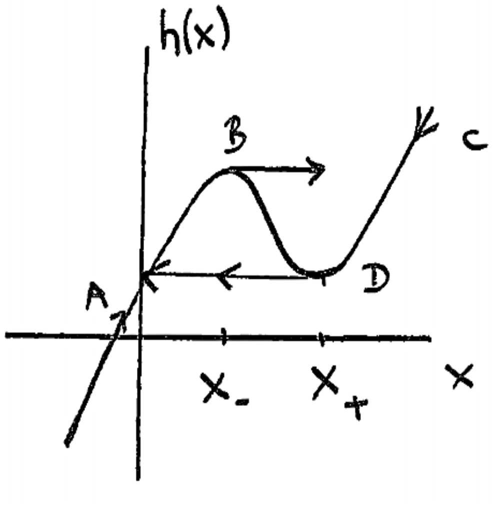
$$2(t_B - t_C) \propto 3 - 2\ln 2.$$

3.3 Extension

Suppose we have the system

$$\epsilon\dot{x} = y - h(x),$$

$$\dot{y} = q(x, y),$$



with $q(x, h(x)) > 0$ if $x < x_-$ and $q(x, h(x)) < 0$ if $x > x_+$. Then periodic oscillations can occur with period

$$P = \int_A^B \frac{h(x)}{q[x, h(x)]}dx + \int_D^C \frac{h'(x)}{|q(x, h(x))|}dx$$

Here we have set $\epsilon \approx 0$ to write $y = h(x)$, so that $dt = \frac{dy}{q} = \frac{dxh'}{q}$ on the RHS of the expression for P.

The Outer solution ($\epsilon = 0$) is $y = h(x)$ with

$$\dot{y} \sim q(x, h(x)) \Rightarrow \dot{x} = \frac{q(x, h(x))}{h'(x)}.$$

The Inner solution: near t_B we can write $t = t_B + \epsilon s$, so that $dt = \epsilon ds$, which gives

$$\begin{aligned} x' &= y - h(x), \\ y' &= \epsilon q. \end{aligned}$$

Therefore at $y \sim y_B$, and $x' = y_B - h(x)$, therefore it follows that $x \to x_C$.