LECTURE 6:

THE KRYLOV-BOGOLIUBOV METHOD OFAVERAGING

Consider the autonomous system

$$\ddot{x} + x = \epsilon f(x, \dot{x}). \tag{2.1}$$

When $\epsilon = 0$, (6.1) reduces to the equation for SHM: $\ddot{x} + x = 0$ with general solution

$$x(t) = A\cos t + B\sin t = a\cos(t + \theta), \tag{2.2}$$

for some constant amplitude a and phase angle $t + \theta$. Then

$$\dot{x} = -a\sin(t + \theta). \tag{2.3}$$

Krylov and Bogoliubov assumed the solution to (6.1) is still (6.2) and (6.3), but now with time-varying a and θ, ie.

$$x(t) = a\cos(t + \theta(t)), \tag{2.4}$$

together with the extra constraint:

$$\dot{x}(t) = -a(t)\sin(t + \theta(t)). \tag{2.5}$$

(Since $a(t)$ and $\theta(t)$ are arbitrary, we can choose any constraint.)

Then differentiating both sides of (6.4) wrt t we obtain

$$\dot{x}(t) = -a\sin(t + \theta)(1 + \dot{\theta}) + \dot{a}\cos(t + \theta),$$

and using (6.5) we get, upon subtraction:

$$\dot{a}\cos(t + \theta) - a\dot{\theta}\sin(t + \theta) = 0. \tag{2.6}$$

Differentiating (6.5) wrt t, we get

$$\ddot{x} = -a\cos(t+\theta)(1+\dot{\theta}) - \dot{a}\sin(t+\theta),$$

so that (6.1) becomes

$$-a\cos(t+\theta)(1+\dot{\theta}) - \dot{a}\sin(t+\theta) + a\cos(t+\theta) = \epsilon f(a\cos(t+\theta), -a\sin(t+\theta)),$$

and it follows that

$$\dot{a}\sin(t+\theta) + a\dot{\theta}\cos(t+\theta) = -\epsilon f(a\cos(t+\theta), -a\sin(t+\theta)). \qquad (2.7)$$

Solving (6.6) and (6.7) for $\dot{a}$ and $\dot{\theta}$, we obtain:

$$\begin{aligned} \dot{a} &= -\epsilon f\Big(a\cos(t+\theta), -a\sin(t+\theta)\Big)\sin(t+\theta), &(2.8a)\\ \dot{\theta} &= -\epsilon\frac{f}{a}\cos(t+\theta). &(2.8b) \end{aligned}$$

Writing $\phi = t+\theta$, so that

$$\dot{\theta} = \dot{\phi} - 1 = -\epsilon\frac{f}{a}\cos(\phi)$$

gives

$$\dot{\phi} = 1 - \frac{\epsilon f}{a}\cos\phi.$$

Thus

$$\begin{aligned} \dot{a} &= -\epsilon f\Big(a\cos\phi, -a\sin\phi\Big)\sin\phi, &(2.9a)\\ \dot{\phi} &= 1 - \epsilon\frac{f}{a}\Big(a\cos\phi, -a\sin\phi\Big)\cos\phi, &(2.9b) \end{aligned}$$

and we obtain a pair of first order ODEs for the evolution of the amplitude $a(t)$ and phase $\phi(t)$. Since both $\dot{a}$ and θ are both $\mathcal{O}(\epsilon)$, a and θ are slowly varying functions of time, changing little over one period of $\tau = 2\pi$.

Because the RHS of (6.9) are periodic functions of θ, we can integrate over $[0, 2\pi]$, taking a and θ to be constant, i.e. we average over one period.

Definition 1. *Define*

$$\langle g(\phi)\rangle = \frac{1}{2\pi}\int_0^{2\pi} g(\phi)d\phi$$

to be the average of $g(\theta)$ *over* 2π.

Taking $\theta \approx constant$, $d\phi = dt + d\theta \approx dt$ for $0 \le \theta \le 2\phi$. When $\phi = 0$, $t = -\theta$; and when $\phi = 2\pi$, $t = 2\pi - \theta$.

Therefore

$$\frac{1}{2\pi}\int_0^{2\pi} g(\phi)d\phi = \frac{1}{2\pi}\int_{-\theta}^{2\phi-\theta} g(t)dt = \frac{1}{\tau}\int_0^{\tau} g(t)dt,$$

where $\tau = 2\pi$, by property of integrals.

Thus, averaging (6.9), we get

$$\begin{aligned} \dot{a} &= -\epsilon\langle f\sin\phi\rangle, && (2.10a)\\ \dot{\phi} &= 1 - \frac{\epsilon}{a}\langle f\cos\phi\rangle, && (2.10b)\\ \dot{\theta} &= -\frac{\epsilon}{a}\langle f\cos\phi\rangle. && (2.10c) \end{aligned}$$

Examples

1. Duffing's Equation We show how to use the Method of Averaging on Duffing's equation. Consider

$$\ddot{x} + x = -\epsilon x^3 \quad \epsilon f(x),$$

where, again $\epsilon \ll 1$. When $\epsilon = 0$, we have a solution $x = a\cos\phi$, so that $x^3 = a^3\cos^3\phi$.

Therefore

$$\begin{aligned} \langle f\sin\phi\rangle &= \langle -x^3\sin\phi\rangle \\ &= -\frac{a^3}{2\pi}\int_0^{2\pi}\cos^3\phi\sin\phi d\phi \\ &= \frac{a^3}{2\pi}\left[\frac{\cos^4\phi}{4}\right]_0^{2\pi} = 0; \end{aligned}$$

and

$$\begin{aligned} \langle f\cos\phi\rangle &= \langle -x^3\cos\phi\rangle \\ &= -\frac{a^3}{2\pi}\int_0^{2\pi}\cos^4\phi d\phi \\ &= -\frac{a^3}{2\pi}\int_0^{2\pi}\left\{\frac{1}{2}(1+\cos 2\phi) - \frac{1}{8}(1-\cos 2\phi)\right\}d\phi \\ &= -\frac{a^3}{2\pi}\left[\frac{3\phi}{8}\right]_0^{2\pi} = -\frac{3a^3}{8}. \end{aligned}$$

Thus it follows from (6.10) that $\dot{a} = 0$ implies that $a = a_0$, a constant, and

$$\dot{\theta} = \frac{3}{8}a^2\epsilon = \frac{3}{8}a_0^2\epsilon,$$

so that

$$\theta = \frac{3}{8}a^2\epsilon t + \theta_0.$$

With initial conditions $\theta(0) = 0$, we can take $\theta_0 = 0$, so that

$$x = a_0\cos(t + \theta) = a_0\cos(t + \frac{3}{8}a_0^2\epsilon t) + \mathcal{O}(\epsilon^2).$$

Thus the amplitude is a_0, the frequency is $\omega = 1 + \epsilon\frac{3a_0^2}{8}$, and the period is $\frac{2\pi}{\omega}$ cf with previous results.

<u>2. Van der Pol equation</u>

Consider

$$\ddot{x} + x = \epsilon(1 - x^2)\dot{x}$$

For $\epsilon = 0$, we take $x = a\cos\phi$, $\dot{x} = -a\sin\phi$. Then

$$\epsilon f(x) = -\epsilon(1 - a^2\cos^2\phi)a\sin\phi.$$

Therefore

$$\begin{aligned}\langle f\sin\phi\rangle &= \langle -(1 - a^2\cos^2\phi)a\sin^2\phi\rangle \\ &= -\frac{1}{2\pi}\int_0^{2\pi}[a\sin^2\phi - a^3\cos^2\phi\sin^2\phi]d\phi \\ &= -\frac{1}{2\pi}\int_0^{2\pi}\Big(\frac{a}{2}[1 - \cos 2\phi] - \frac{a^3}{8}[1 - \cos 4\phi]\Big)d\phi \\ &= -\frac{a}{2}\Big(1 - \frac{a^2}{4}\Big).\end{aligned}$$

Hence

$$\dot{a} = -\epsilon\langle f\sin\phi\rangle = \epsilon\frac{a}{2}(1 - \frac{a^2}{4}).$$

Also

$$\begin{aligned}\langle f\cos\phi\rangle &= \langle -(1 - a^2\cos^2\phi)a\sin\phi\cos\phi\rangle \\ &= -\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{a}{2}\sin 2\phi - a^3\cos^3\phi\sin\phi\right]d\phi \\ &= 0.\end{aligned}$$

Hence we have

$$\dot{\theta} = 0 \Rightarrow \theta = \theta_0,$$

a constant to $\mathcal{O}(\epsilon)$ and

$$\dot{a} = \epsilon \frac{a}{2}\left[1 - \frac{a^2}{4}\right].$$

It follows that

$$a\dot{a} = \epsilon \frac{a^2}{2}\left[1 - \frac{a^2}{4}\right].$$

Let $r = a^2$. Then $\dot{r} = \epsilon r(1 - \frac{r}{4})$, and integrating over time we obtain

$$r = a^2 = \frac{4}{1 + \left(\frac{4}{a_0^2} - 1\right)e^{-\epsilon t}}.$$

Therefore, as $t \to \infty$ we have $a^2 \to 4 \Rightarrow a \to 2$ and we obtain the Van der Pol small amplitude limit cycle as before.