LECTURE 4: HOPF BIFURCATIONS

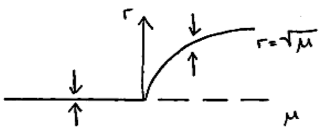
Suppose $\lambda = \pm i\omega$ at $\mu = \mu_c$. Consider the (r, θ) form:

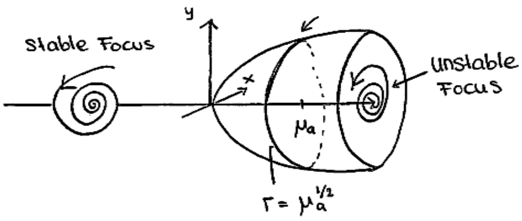
$$\dot{r} = \mu r - r^3,$$

$$\dot{\theta} = \omega,$$

where $\mu > 0$ and $\omega > 0$.

The '$\dot{r}$' equation 'decouples' from the '$\dot{\theta}$' equation and (by the pitchfork analysis) we have





The $\dot{\theta}$ equation provides an anticlockwise rotation. The solution, for $\mu > 0$ is an envelope of periodic solutions or limit cycles, which increase in amplitude like $\mu^{1/2}$ with frequency $\approx \omega$.

This is a supercritical Hopf Bifurcation (reverse the sign of r^3 for a subcritical Hopf bifurcation). The period τ of the limit cycle is $\sim \frac{2\pi}{\omega} + h.o.t.$

4.1 Global Bifurcations of Limit-Cycles

Global bifurcations can destroy limit cycles.

4.2 Saddle-Node Bifurcations of Limit Cycles

Consider the following system:

$$\dot{r} = \mu r + r^3 - r^5 = f,$$

$$\dot{\theta} = \omega + br^2,$$

for $\mu < 0$ and $r \geq 0$. Since there is no θ in the $\dot{r}$ equation, the two equations decouple and we can consider the $\dot{r}$ equation alone. The solution for r will then just modify the frequency by a quadratic correction term. Thus $\dot{r} = 0$ implies:

$$r = 0 \quad \text{or} \quad \mu + r^2 - r^4 = 0, \tag{4.1}$$

so that

$$r^2 = \frac{1}{2} \pm \frac{1}{2}[1 + 4\mu]^{1/2}.$$

Thus when

$$\mu < -\frac{1}{4}, \quad \text{there are no real roots,}$$

$$\mu_c = -\frac{1}{4}, \quad r^2 = 1/2 \text{ (there are two equal roots),}$$

$$\mu > -\frac{1}{4}, \quad \text{there are two distinct roots for} \quad r^2.$$

Stability is determined by

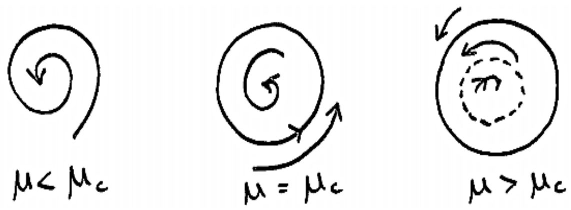
$$f_r = \mu + 3r^2 - 5r^4,$$

For $r = 0$, $f_r = \mu$, and since $\mu < 0$, we have stability.
For $r \neq 0$, we can use the above quartic equation for r to write $f_r = -2(r^2+2\mu)$. When $\mu = \mu_c$, $r^2 = 1/2$, $\mu_c = -1/4$, and $f_r = 0$, and we have a coalescence of two limit cycles.

For $\mu \geq \mu_c$:

$$\Lambda = f_r = -2\left[\frac{1}{2} + 2\mu \pm \frac{1}{2}(1+4\mu)^{1/2}\right] = \left[(1+4\mu) \pm (1+4\mu)^{1/2}\right].$$

The $+$ solution has $\Lambda < 0$ and is therefore stable, while the $-$ solution has $\Lambda > 0$ and is therefore unstable. We have a saddle-node bifurcation of limit cycles at $\mu = \mu_c$



4.3 Homoclinic Bifurcations

Consider the system

$$\dot{x} = y = \qquad f$$

$$\dot{y} = \mu y - x^2 + xy = g$$

Equilibria are : $y = 0$, $x = 0, 1$, with Jacobian matrix:

$$J = \begin{bmatrix} 0 & 1 \\ 1 - 2x + y & \mu + x \end{bmatrix}.$$

Characteristic equation is :

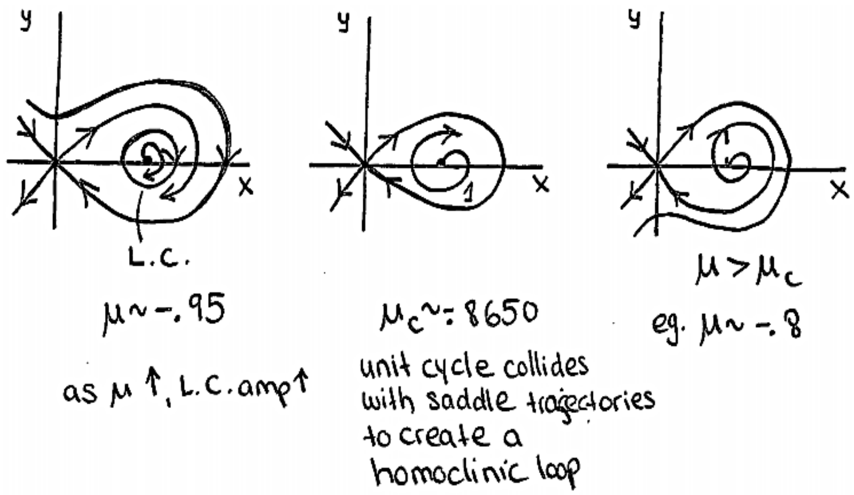
$$\lambda^2 - (\mu + x_e)\lambda - (1 - 2x_e + y_e) = 0.$$

For $(0,0)$, $\lambda_0^2 - \mu\lambda_0 - 1 = 0$, so that $\lambda_0 = \frac{\mu}{2} \pm \frac{1}{2}\left[\mu^2 + 4\right]^{1/2}$.

For $(1,0)$, $\lambda_1^2 - (\mu+1)\lambda_1 + 1 = 0$, which gives $\lambda_1 = \frac{1}{2}((\mu+1) \pm \left[(\mu+1)^2 - 4\right]^{1/2})$.

Since $\mu^2 + 4 > \mu$ for all μ, $(0,0)$ is a saddle for all μ.

$(1,0)$ has a Hopf Bifurcation at $\mu = -1$ since $\lambda_1 = \pm i$, with the transversality condition: $\frac{\partial \lambda_1^r}{\partial \mu} = \frac{1}{2} \neq 0$.
For $(\mu+1)^2 = 4$, $\mu = 1$ or $\mu = -3$.
When $\mu = -3$, $\lambda_1 = -1$, and we have a stable node. For $-3 < \mu < -1$, we have a stable spiral and an unstable spiral for $-1 < \mu < 1$. For $\mu > 1$ we have an unstable node.
Numerics show:



4.4 Josephson Junction

Consider the system

$$\dot{\phi} = y$$

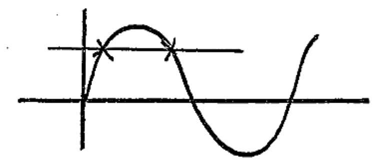
$$\dot{y} = I - \sin(\phi) - \alpha y,$$

where $\alpha > 0$ is damping and $I \geq 0$ is applied current. Equilibrium states are given by

$$y = 0, \quad \sin \phi_* = I \quad \text{provided } |I| \leq 1.$$

<u>Stability:</u>

$$J = \begin{bmatrix} 0 & 1 \\ -\cos\phi_* & \alpha \end{bmatrix} \quad \Rightarrow \lambda^2 + \alpha\lambda + \cos\phi_* = 0,$$



Therefore

$$\lambda = \frac{-\alpha}{2} \pm \frac{1}{2}[\alpha^2 - 4\cos\phi_*]^{1/2},$$

and there are two equilibria determined from

$$\cos\phi_* = [1 - \sin^2\phi_*]^{1/2} = [1 - I^2]^{1/2},$$

so that

$$\lambda_\pm = -\frac{\alpha}{2} \pm \frac{1}{2}[\alpha^2 - 4(1 - I^2)^{1/2}]^{1/2}.$$

when $I = 1$, $\lambda_\pm + 0, -\alpha$ and we have a saddle-node bifurcation. For I close to 1, $\lambda_\pm < 0$ and we have stable node.