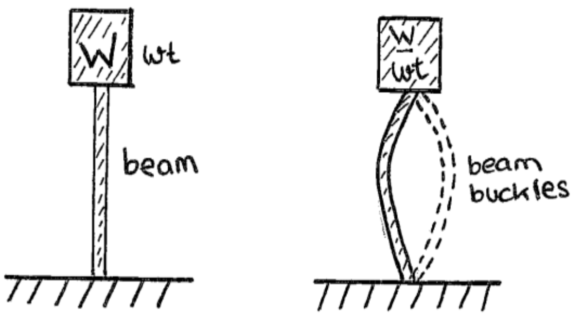
LECTURE 3: BIFURCATION THEORY

As well as finding fixed points and classifying them, we are also interested in what happens to the stability of the equilibria as some parameters vary. Suppose the system

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \mu) \tag{3.1}$$

now depends upon some parameter μ. Then the fixed points and their stabilities will also depend upon μ.

Consider, for example, a beam, supporting a weight W.



For a sufficiently large W, the beam's vertical position becomes unstable and buckling occurs. This is an example of a bifurcation, namely a qualitative change in the behaviour of a system as a parameter varies. Here the weight W plays the role of the parameter.

3.1 Elementary Bifurcations

A bifurcation occurs when a given equilibrium becomes unstable and a new (stable) state is established. In terms of eigenvalues of the Jacobian matrix J: one or more eigenvalues cross the Imaginary axis.

When one parameter varies we have a codimension-one bifurcation. There are four basic types of codimension-one bifurcations, three types when an eigenvalue passes through zero: the Saddle-Node, the Transcritical and the Pitchfork bifurcation. The fourth type, a Hopf bifurcation, occurs when a pair of complex conjugate eigenvalues crosses the Imaginary axis with a nonzero speed.

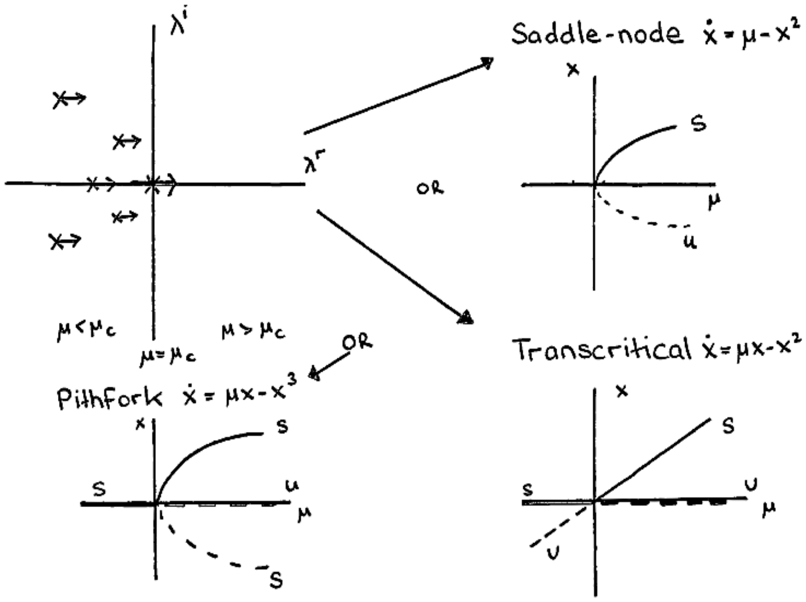


Figure 1: Three types of codimension-one bifurcations: Saddle-Node, Transcritical and Pitchfork bifurcations, where $\lambda = 0$ at $\mu = \mu_c$

N.B. If sign of the nonlinear term is negative, we have a supercritical bifurcation.

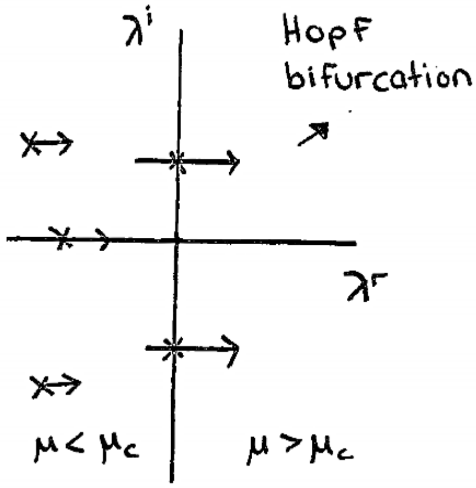


Figure 2: Hopf bifurcation, where $\lambda = \pm i\omega$ at $\mu = \mu_c$

Thus for the Hopf bifurcation

$$\lambda = \pm i\omega \quad \text{at } \mu = \mu_c,$$

together with the Transversality Condition:

$$\frac{\partial \lambda^r}{\partial \mu} \neq 0 \quad \text{at } \mu = \mu_c$$

which ensures that the pair of complex conjugate eigenvalues cross clearly from $\mu < \mu_c$ to $\mu > \mu_c$ without stopping or turning back, etc.
The normal form for a Hopf Bifurcation is either

$$\dot{x} = -\omega y + x[\mu - (x^2 + y^2)], \tag{3.2a}$$
$$\dot{y} = \omega x + y[\mu - (x^2 + y^2)], \tag{3.2b}$$

in cartesian coordinates or

$$\dot{r} = \mu r - r^3, \tag{3.3a}$$
$$\dot{\theta} = \omega, \tag{3.3b}$$

in polar coordinates, where $x = r\cos\theta$ and $y = r\sin\theta$.

3.2 Saddle-Node Bifurcation

The normal form for a saddle-node bifurcation is

$$\dot{x} = \mu - x^2 = f(x, \mu), \quad \dot{y} = -y. \tag{3.4}$$

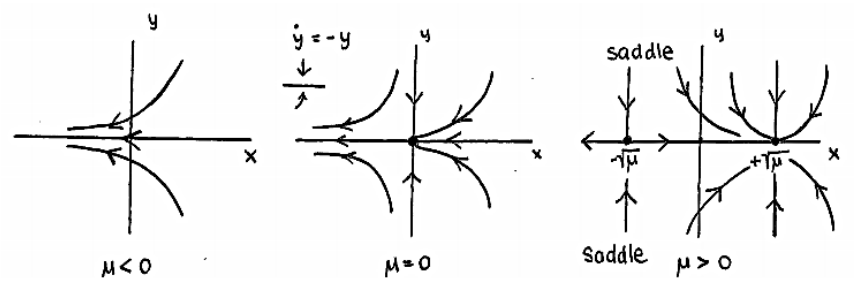
Since the $\dot{x}$ and $\dot{y}$ equations decouple, the $\dot{x}$ equations give the bifurcations, while the $\dot{y}$ equation gives exponential damping in y.
Fixed points are given by $x = \pm\sqrt{x}$, provided $\mu \geq 0$. We have three situations:

(i) $\mu < 0$: no fixed points,

(ii) $\mu = 0$: 2 equal fixed points $x = 0$,

(iii) $\mu > 0$: $x_e = \pm\sqrt{\mu}$, two distinct fixed points.

Stability is determined by the eigenvalue $\lambda = f_x(x_e, \mu_e) = -2x_e$.
Therefore For $x_e = +\sqrt{\mu}$, $\lambda < 0$, which gives stability, whereas for $x_e = -\sqrt{\mu}$, $\lambda > 0$ which gives instability. We illustrate this below.



As an example, consider the system

$$\dot{x} = -ax + y = f,$$

$$\dot{y} = \frac{x^2}{1+x^2} - by = g,$$

where $a, b > 0$.

The fixed points are $(0,0)$ or (x_e, y_e) where $y_e = ax_e$ and

$$x_{e\pm} = \frac{1}{2ab} \pm \frac{1}{2ab}[1 - 4a^2b^2]^{1/2}, \quad \text{for} \quad 4a^2b^2 < 1.$$

When $1 < 4a^2b^2$ only the fixed point $(0,0)$ exists. When $1 = 4a^2b^2$, we have three fixed points: $(0,0)$, $(1,a)$ (twice). Finally when $1 > 4a^2b^2$ we have three distinct fixed points: $(0,0)$, $(x_{e\pm}, ax_{e\pm})$. The Jacobian matrix is given by

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -a & 1 \\ \frac{2x}{(1+x)^2} & -b \end{bmatrix},$$

so that the characteristic equation becomes:

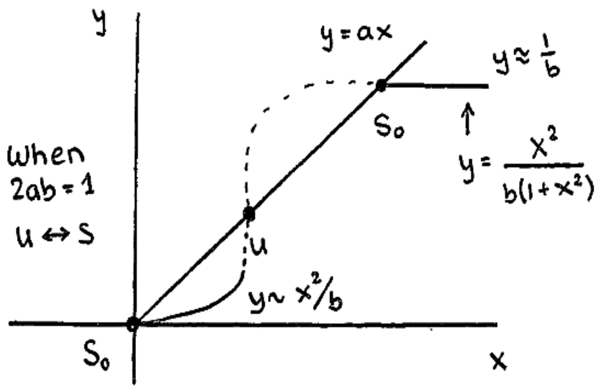
$$\Lambda^2 + (a+b)\Lambda + ab - \frac{2x}{(1+x^2)^2} = 0.$$

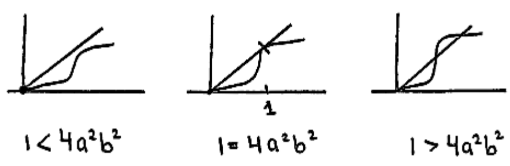
For $(0,0)$, we have

$$\Lambda^2 + (a+b)\Lambda + ab = 0,$$

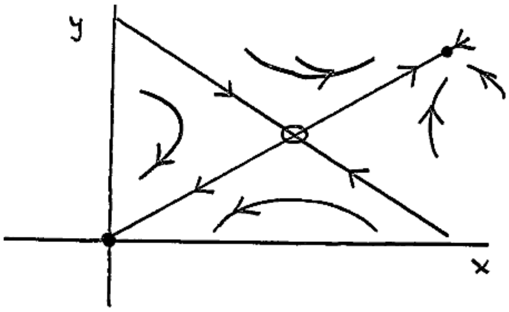
so that $\Lambda = -a, -b$ and $(0,0)$ is a stable node since $a, b > 0$.

Nullclines are given by $\dot{x} = 0$ so that $y = -ax$, and $\dot{y} = 0$, which gives $y = -\frac{x^2}{b(1+x^2)}$.





When $x << 1$, we have $y \sim \frac{x^2}{b}$, while for $x >> 1$, $y \sim \frac{1}{b}$. The figure shows the equilibrium states as intersections of the nullclines when a and b vary. We see that when $2ab = 1$, the nontrivial stable and stable fixed points coalesce in a saddle-node bifurcation.



3.3 Pitchfork Bifurcation

The normal form for a pitchfork bifurcation is

$$\dot{x} = \mu \; - x^3 \quad = F(x, \mu). \tag{3.5}$$

Since this equation is invariant under the transformation $x \to -x$, the pitchfork bifurcation is reflectionally symmetric.

The equilibrium states are given by $x = 0 \; \forall \mu$, and $x^2 = \mu$ for $\mu \geq 0$. Stability is given by $\lambda = f' = \mu - 3x^2$, evaluated at each equilibrium state in turn. We see that $x_0 = 0$ is stable for $\mu < 0$, but unstable for $\mu > 0$, whereas $x_e = \pm\mu$ are always stable. We have a supercritical pitchfork bifurcation.

Example Consider

$$\dot{x} = \mu x + y + \sin x = f$$

$$\dot{y} = x - y = g$$

The trivial equilibrium $x = y = 0$ is am equilibrium for all values of μ, with stability determined from

$$J_{\underline{0}} = \begin{bmatrix} \mu + \cos x & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \mu + 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

This yields the characteristic equation:

$$\Lambda^2 - \mu\Lambda - (\mu + 2) = 0.$$

Bifurcations occur when $\Lambda = 0$, which gives $\mu_c = -2$. Since

$$\Lambda = \frac{\mu}{2} \pm \frac{1}{2}[\mu^2 + 4(\mu + 2)],$$

(i) for $\mu < -2$:

$$[\mu^2 + 4(\mu + 2)]^{1/2} < |\mu|^2$$

and we have a stable node.

(ii) For $\mu > -2$:

$$[\mu^2 + 4(\mu + 2)]^{1/2} > |\mu|^2$$

and we have a saddle.

Close to $x = 0$, we expand the rhs of the $\dot{x}$ equation to get

$$\mu x + y + x - \frac{x^3}{3!} \approx 0$$

$$x = y$$

so that provided $\mu > -2$ $(\mu + 2) \approx \frac{x^2}{6}$ or $x \approx \pm\sqrt{6(\mu + 2)}$.
Then

$$J \approx \begin{bmatrix} \mu + 1 - \frac{x^2}{2} & 1 \\ 1 & -1 \end{bmatrix} \approx \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},$$

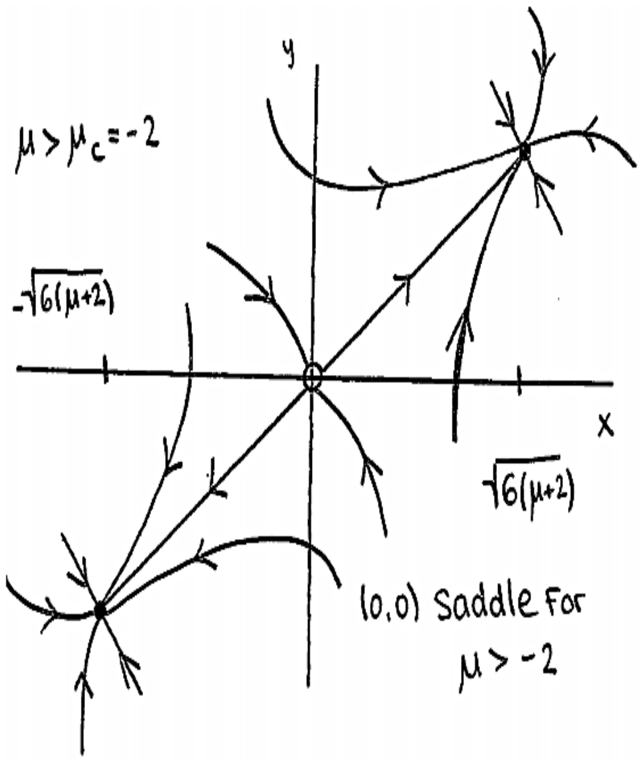
evaluated at $\mu_c \approx -2$ with $x_e^2 = 6(\mu + 2)$.

The characteristic equation is

$$\lambda^2 + 2\lambda = 0, \quad \text{so that} \quad \lambda = 0, -2$$

and we have marginal stability.

The eigenvectors of J are $[1, 1]^T$ for $\lambda = 0$ and $[1, -1]^T$ for $\lambda = -2$.



[In general we have

$$\begin{bmatrix} \mu+1-\frac{x^2}{2} & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \mu+1-3(\mu+2) & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2\mu-5 & 1 \\ 1 & -1 \end{bmatrix}.$$

Hence

$$\Lambda^2 + (2\mu+6)\Lambda + (2\mu+4) = 0,$$

so that

$$\Lambda = -(\mu+3) \pm \left[(\mu+3)^2 - 2(\mu+2)\right]^{1/2}$$

and we have two stable nodes for $\mu \geq -\mu_c$