Lecture 1: Models leading to Laplace's equation

Laplace's equation crops up in a variety of practically motivated models. Here are three.

Steady heat flow

Fourier's law of heat conduction states that the heat flux in a homogeneous isotropic medium D of constant thermal conductivity k is

$$\mathbf{q} = -k\boldsymbol{\nabla}u,$$

where u is the temperature. When the temperature is time-independent, conservation of energy in the form $\boldsymbol{\nabla}\cdot\mathbf{q} = 0$ immediately leads to Laplace's equation for u:

$$\boldsymbol{\nabla}^2 u = 0 \qquad \text{in } D.$$

In two space dimensions this means that u can be written as the real part of a holomorphic function $f(z)$, $z = x + \mathrm{i}y$. If $f(z) = u + \mathrm{i}v$, the conjugate function v has a physical interpretation. Take two points z_0, z_1 in D and a curve Γ joining them, then the heat flux across the curve is

$$\int_\Gamma \mathbf{q}\cdot\mathbf{n}\,\mathrm{d}s = -k\int_\Gamma \frac{\partial u}{\partial n}\,\mathrm{d}s = -k\int_\Gamma \frac{\partial v}{\partial s}\,\mathrm{d}s = -k\,[v]_{z_0}^{z_1},$$

where we have used the Cauchy–Riemann equations to switch from u to v. Thus v is a 'heat streamfunction'.

Typical boundary conditions for u are either that u is given on ∂D, or that $\partial u/\partial n$ is given there; an insulated boundary corresponds to $\partial u/\partial n = 0$.

Inviscid fluid flow

The simplest model for a fluid is that it is inviscid and incompressible (fortunately this is a remarkably accurate model in many circumstances). An incompressible fluid of constant mass density ρ has a velocity field $\mathbf{u}$ that satisfies

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0,$$

which is simply conservation of fluid mass applied to a small fixed control volume (just like conservation of energy for heat flow). In order to write down an equation of motion for the fluid we need to apply ' mass $\times$ acceleration = force' in a frame moving with a fluid particle. The acceleration of the fluid in this frame is

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}$$

(just the chain rule, changing to coordinates in a frame moving with velocity $\mathbf{u}$). The force on a fluid particle has two sources: gravity and the fluid's internal forces. The latter are assumed to be solely due to the internal pressure of the fluid, which acts isotropically (that is, there are no internal frictional forces, as would be caused by viscosity). Putting these together to use Newton's law on a small material volume V of the fluid (one that moves with velocity $\mathbf{u}$), we have

$$\int_V \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\,\mathrm{d}V = \int_{\partial V} -p\mathbf{n}\,\mathrm{d}S + \int_V \rho\mathbf{g}\,\mathrm{d}V = \int_V -\boldsymbol{\nabla}p + \rho\mathbf{g}\,\mathrm{d}V,$$

where $\mathbf{g}$ is the acceleration due to gravity. As V is arbitrary, this leads to Euler's equations

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}p + \rho\mathbf{g}, \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0.$$

These horribly nonlinear equations have a quite remarkable consequence in the form of Kelvin's theorem. This says that if we take a small closed curve Γ of fluid particles and follow its evolution, then the *circulation* round Γ,

$$\oint_\Gamma \mathbf{u}\cdot\mathrm{d}\mathbf{s},$$

is a constant. In particular, if Γ starts with zero circulation, its circulation is always zero. Now by Stokes' theorem, the circulation is also equal to

$$\iint_S \nabla \wedge \mathbf{u} \cdot \mathrm{d}\mathbf{S},$$

where S spans Γ. If all fluid particles start in a region where the vorticity $\boldsymbol{\omega} = \nabla \wedge \mathbf{u} = \mathbf{0}$, for example in a uniform flow 'far upstream', then we have the result that $\boldsymbol{\omega} \equiv \mathbf{0}$ throughout the flow. Such a flow is called *irrotational*.[1]

The condition $\nabla \wedge \mathbf{u} = \mathbf{0}$ means that in an irrotational flow $\mathbf{u}$ can be written as $\mathbf{u} = \nabla\phi$, where, because $\nabla \cdot \mathbf{u} = 0$,

$$\nabla^2\phi = 0.$$

The function ϕ is called the velocity potential. Replacing $\mathbf{u}$ by $\nabla\phi$, we find that the Euler equations reduce to

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + \Omega\right) = \mathbf{0}, \qquad \nabla^2\phi = 0,$$

where $\mathbf{g} = -\nabla\Omega$. The first of these can be integrated to give

$$\frac{\partial\phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + \Omega = F(t),$$

where $F(t)$ is an arbitrary function of time (which can often be incorporated into ϕ). This is *Bernoulli's equation* representing (in steady flow) the distribution of energy between kinetic, potential and 'work done by the pressure'.

Let us now (and for the rest of this course) consider only steady two-dimensional flows with zero gravity ($\Omega = 0$). We then have $\mathbf{u} = (u_1, u_2, 0)$, $\phi = \phi(x, y)$ and

$$u_1 = \frac{\partial\phi}{\partial x}, \qquad u_2 = \frac{\partial\phi}{\partial y}.$$

However, because $\nabla \wedge \mathbf{u} = \mathbf{0}$, we also have a function ψ such that

$$u_1 = \frac{\partial\psi}{\partial y}, \qquad u_2 = -\frac{\partial\psi}{\partial x}.$$

Here ψ is the *streamfunction* for the flow and, as in the heat flow example above, it measures the mass flux. It is constant on *streamlines* which, in steady flow (only) are the paths followed by fluid particles. The two equations above taken together are the Cauchy–Riemann equations for the holomorphic function[2]

$$w(z) = \phi + \mathrm{i}\psi$$

is called the *complex potential* for the flow.

We also note that the conditions

$$\nabla \cdot \mathbf{u} = 0, \qquad \nabla \wedge \mathbf{u} = \mathbf{0}.$$

say that

$$\frac{\partial u_1}{\partial x} = -\frac{\partial u_2}{\partial y}, \qquad \frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial x}.$$

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[1] In two space dimensions, an alternative to using Kelvin's theorem is as follows. First write Euler's equations in the form

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \boldsymbol{\omega} \wedge \mathbf{u}\right) = -\nabla\left(p + \frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) + \rho\mathbf{g},$$

then take the curl to yield (by a standard vector identity)

$$\rho\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \boldsymbol{\omega} \cdot \nabla\mathbf{u}.$$

In two space dimensions, $\mathbf{u}$ and $\boldsymbol{\omega} = \nabla \wedge \mathbf{u}$ are orthogonal; hence the right-hand side vanishes, and this equation says that $\boldsymbol{\omega}$ is conserved following a fluid particle. Hence, flows that start life with zero vorticity (*e.g.* from a state of rest, or steady flow beginning with a uniform stream) remain irrotational.

[2] We are assuming steady flow only for this course; in unsteady flow w is a function of time t as well (but still holomorphic).

These are the Cauchy–Riemann equations for the function $u - iv$, which is a holomorphic function of $z = x + iy$ called the complex velocity. Clearly,

$$u_1 - iu_2 = \frac{dw}{dz}.$$

Bernoulli's equation then reads

$$\frac{1}{2}\left|\frac{dw}{dz}\right|^2 + \frac{p}{\rho} = \text{constant}$$

We shall consider two types of boundary to the fluid domain. At a solid boundary, the normal component of $\mathbf{u}$ is continuous; this usually means that $\mathbf{u} \cdot \mathbf{n} = 0$ on such a boundary[3] and from the Cauchy–Riemann equations this means that on such a boundary ψ is constant, so the boundary is a streamline. Of course, ψ may have different values on different solid boundaries.

The second type of boundary we consider is a *free surface* between the fluid and (say) air at constant pressure. Although we can say immediately that, in steady flow, such a boundary is a streamline, on which ψ is constant, its location is *a priori* unknown and so we need an extra condition to fix it. This is the Bernoulli condition, that the pressure at the free surface is equal to the external pressure (which is constant); hence on a free surface in steady flow,

$$\left|\frac{dw}{dz}\right|^2 = \text{constant},$$

the constant being fixed by the details of the flow.

Electrostatics

The third model that is often seen in the context of Laplace's equation in two dimensions is that of electrostatics. It was established by Coulomb that a small movable charge of strength q a distance r from a fixed charge of strength Q feels a force proportional to qQ/r^2: this is the inverse square law, just as in Newtonian gravitation (except for the possibility of charges repelling, which appears not to happen for masses). This tells us that a fixed charge generates an electric field $\mathbf{E}$ field which is proportional to the gradient of the potential Q/r, and the force on the movable charge is then $q\mathbf{E}$. As we know, Q/r is a solution of Laplace's equation, and what we have altogether is: *any collection of charges in an otherwise electrically neutral medium (such as a vacuum) generates a potential ϕ which is a solution of Laplace's equation except at the charges.* This ϕ is the usual voltage we talk about in the context of batteries, mains electricity, lightning etc.

A common and useful boundary condition for ϕ is that it is constant on a good conductor (*e.g.* a metal), as any potential differences within the metal would drive a current (of charge) that would rapidly force them to zero. Thus a canonical problem is to determine the potential between two perfect conductors each of which is held at a different potential (such an arrangement is often called a capacitor).

Review of core complex analysis

This section is a summary and by no means a complete treatment.

Notation

Throughout, overbars denote complex conjugate; $z = x + iy$, $\overline{z} = x - iy$; a region in the complex plane, usually denoted D, is a an open, path-connected subset of $\mathbb{C}$, simply-connected unless stated otherwise, and its boundary is ∂D; a contour Γ is a piecewise continuously differentiable, simple path in $\mathbb{C}$ with the positive (anti-clockwise) orientation (a Jordan contour), closed unless stated otherwise; $D(a; R)$ is the disc centre a and radius R.

Lecture 1: Models leading to Laplace's equation

Differentiable, holomorphic

It all begins with the innocuous definition that a function $f(z)$ of the complex variable z is *differentiable* at the point z if

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} \tag{1}$$

exists however $h \to 0$; and $f(z)$ is *holomorphic* (some people use *analytic*) in a region D if it is differentiable at each point of D.

Writing $f = u + \mathrm{i}v$ where u and v are real, then taking h first real then imaginary and setting the two resulting values of $f'(z)$ equal, we find the **Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

relating the real and imaginary parts of $f(z)$ wherever $f(z)$ is differentiable.

It is often said that a holomorphic function is one that in independent of $\overline{z}$. There is a lot more to the whole definition of complex differentiability than meets the eye, because the two independent variables x and y only occur in it through the single combination $x + \mathrm{i}y$. If one were to think of a general function $G(x, y)$, it could equivalently be viewed as a function $g(z, \overline{z})$ using the one-to-one correspondence between $(z, \overline{z})$ and (x, y). One might then, at least formally, form $\partial g/\partial z$ by perturbing z to $z + h_1$ while keeping $\overline{z}$ fixed, that is computing

$$\lim_{h_1\to 0}\frac{g(z+h_1,\overline{z})-g(z,\overline{z})}{h_1}$$

and likewise find $\partial g/\partial\overline{z}$ by perturbing $\overline{z}$ to $\overline{z}+h_2$ while keeping z fixed. This is apparently inconsistent, because if we perturb z to $z+h_1$ then we must simultaneously perturb $\overline{z}$ to $\overline{z}+\overline{h_1}$, and so $\overline{z}$ cannot be kept fixed. However, an alternative formal definition of the operators $\partial/\partial z$ and $\partial/\partial\overline{z}$ is motivated by the chain rule calculation

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right),$$
$$\frac{\partial}{\partial \overline{z}} = \frac{\partial x}{\partial \overline{z}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \overline{z}}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right),$$

in which z and $\overline{z}$ are treaded as independent variables. From these and the Cauchy–Riemann equations it follows immediately (exercise 1 of sheet 1) that if f is differentiable then

$$\frac{\partial f}{\partial \overline{z}} = 0:$$

this is the sense in which we say that a holomorphic function of z is independent of $\overline{z}$.

The Cauchy–Riemann equations can also be used to show that, if $f(z)$ is holomorphic, so is

$$\overline{f}(z) = \overline{f(\overline{z})}$$

(note the double conjugation). The process of generating the second holomorphic function $\overline{f}(z)$ from $f(z)$ is called *Schwarz reflection*.

Cross-differentiating the Cauchy–Riemann equations shows that u and v are solutions of Laplace's equation (ie they are harmonic functions):

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

This is enormously important in applications, because Laplace's equation arises very frequently, and we can use complex functions to solve boundary value problems for it (in two dimensions, at least).

This is well known; much less well known is that, given a harmonic function $u(x, y)$, the holomorphic function of which it is the real part is

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2\mathrm{i}}\right) - u(0,0) + \text{imaginary constant}.$$

Lecture 1: Models leading to Laplace's equation

This is established by first noting that

$$2u(x,y) = f(x+\mathrm{i}y) + \overline{f}(x-\mathrm{i}y),$$

where $\overline{f}(z)$ is again the holomorphic function $\overline{f(\overline{z})}$. Then we do a daring thing: we assume that we can replace x and y by complex variables (in effect, going into two complex dimensions). In particular, we put $x = z/2$ and $y = z/2\mathrm{i}$, to get

$$2u\left(\frac{z}{2}, \frac{z}{2\mathrm{i}}\right) = f(z) + f(0),$$

and the result follows.

Integrals

The integral of a function of z (or of both z and $\overline{z}$) along a curve Γ, which may be open or closed, is defined parametrically in the obvious way. The following is useful:

Green's (Stokes, divergence) theorem. If $g(z,\overline{z})$ has continuous first partial derivatives in a region D enclosed by a simple closed contour Γ, then

$$\oint_\Gamma g(z,\overline{z})\,\mathrm{d}z = 2\mathrm{i}\iint_D \frac{\partial g}{\partial \overline{z}}\,\mathrm{d}x\mathrm{d}y.$$

This is just the usual "$\oint_\Gamma u\,\mathrm{d}x + v\,\mathrm{d}y = \iint_D v_x - u_y\,\mathrm{d}x\mathrm{d}y$" version of Green's theorem restated in terms of z and $\overline{z}$.

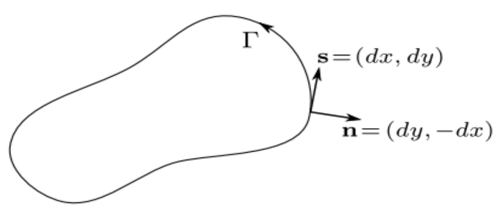

Figure 1: Orientation of $\mathbf{n}$ and $\mathbf{s}$ on Γ

It is also useful to have a way of estimating integrals:

$$\left|\int_\Gamma f(z)\,\mathrm{d}z\right| \le \mathrm{length}(\Gamma) \times \sup_{z\in\Gamma} |f(z)|\,.$$

If Γ is closed, the sup is attained and becomes a max.

Cauchy's theorem and path independence

Having defined integrals of a function of z along a curve by parametrising the curve, we can state the mainspring of complex analysis, Cauchy's theorem:

If a function $f(z)$ is holomorphic within, and continuous on, on a simple curve Γ, then

$$\int_\Gamma f(z)\,dz = 0.$$

This theorem is not at all mysterious. It is, in essence, simply a combination of Green's theorem and the fact that $f(z)$ is independent of $\overline{z}$:

$$\int_\Gamma f(z)\,dz = 2\mathrm{i}\iint_D \frac{\partial f}{\partial \overline{z}}\,\mathrm{d}x\mathrm{d}y = 0.$$

Nice try; no cigar. There is a hidden, unstated assumption that $f(z)$ has continuous first partial derivatives, and it is not, at this stage, known that it does (it will be, soon). The correct proof, due to Goursat, is technical and

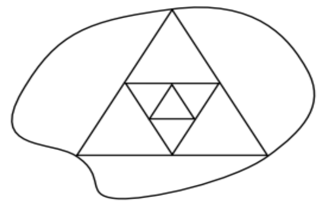

Figure 2: Triangulation of D.

based on estimating the integral over a finer and finer triangulation of D. The usual statement of the theorem takes $f(z)$ to be holomorphic on Γ, rather than just continuous, because this make the theorem easier to prove. Our statement is more powerful; for example, it eliminates the need for some indentations in contour integration (eg when integrating $z^{\frac{1}{2}}/(1+z^2)$ near the origin). Cauchy's own proof needed the additional assumption that $f'(z)$ is continuous.

It is an immediate consequence of Cauchy's theorem that if Γ_1 and Γ_2 are two curves joining the point z_0 to another point z_1, and if $f(z)$ is holomorphic in a region containing Γ_1, Γ_2 and the region between them, then

$$\int_{\Gamma_1} f(z)\,\mathrm{d}z = \int_{\Gamma_2} f(z)\,\mathrm{d}z,$$

so that the integral is path-independent. This is often stated as the **deformation theorem**: if one contour Γ_1 can be deformed smoothly into another one Γ_2 while crossing only points at which $f(z)$ is holomorphic, then the integral of $f(z)$ along Γ_1 is equal to the integral along Γ_2. It also allows us to define an anti-derivative (primitive) of $f(z)$ by the prescription

$$F(z) = \int_{z_0}^{z} f(t)\,\mathrm{d}t,$$

provided that we do so in a simply connected region within which $f(z)$ is holomorphic, as the contour of integration is immaterial.

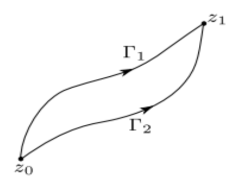

Figure 3: Two paths from z_0 to z_1.

Cauchy's theorem has a partial converse:

Morera's theorem. *If $f(z)$ is continuous in D and $\oint_\Gamma f(z)\mathrm{d}z = 0$ for all simple closed Γ in D, then $f(z)$ is holomorphic in D.*

Morera's theorem has its uses (for example, to prove that the limit of a uniformly convergent sequence of holomorphic functions is holomorphic), but there are better things to come.

Cauchy's integral formula

Take a simple closed contour Γ, and let $f(z)$ be holomorphic on Γ and inside it. Then values of $f(z)$ on Γ determine its values at all points within Γ as well, via **Cauchy's integral formula**: for all z within Γ,

$$f(z) = \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \frac{f(t)}{t-z}\,\mathrm{d}t.$$

The proof is simple, by deforming the contour to a small circle surrounding z and adding and subtracting $f(z)$:

$$\oint_{\Gamma} \frac{f(t)}{t-z}\,\mathrm{d}t. = \oint_{|t-z|=\epsilon} \frac{f(t)}{t-z}\,\mathrm{d}t = \oint_{|t-z|=\epsilon} \frac{f(z)}{t-z}\,\mathrm{d}t + \oint_{|t-z|=\epsilon} \frac{f(t)-f(z)}{t-z}\,\mathrm{d}t;$$

the first integral on the right is equal to $2\pi\mathrm{i}f(z)$ and the second vanishes as $\epsilon \to 0$ by continuity of f.

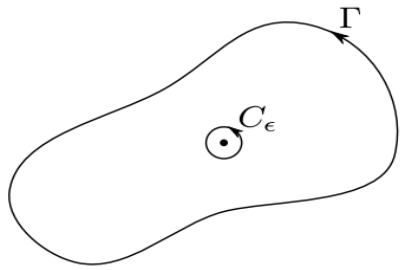

Figure 4: Deformation of Γ to $C_\epsilon = \{z \mid |t-z| = \epsilon\}$.

It is an interesting sidetrack to combine Cauchy's integral formula with the complex Green's theorem. Recalling that $\partial f/\partial \overline{z} = 0$,

$$\begin{aligned} f(z) &= \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma} \frac{f(t)}{t-z}\,\mathrm{d}t. \\ &= \frac{1}{\pi} \iint_D \frac{\partial}{\partial \overline{t}} \frac{f(t)}{t-z}\,\mathrm{d}A. \\ &= \frac{1}{\pi} \iint_D f(t) \frac{\partial}{\partial \overline{t}} \frac{1}{t-z}\,\mathrm{d}A, \end{aligned}$$

where we don't set

$$\frac{1}{\pi}\frac{\partial}{\partial \overline{t}}\frac{1}{t-z} = 0$$

because of the singularity at $t = z$. In fact this quantity is the complex delta function, because when a holomorphic function $f(z)$ is multiplied by it and integrated over a region containing the origin, the result is $f(0)$.

Infinite differentiability!

Given that

$$f(z) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma} \frac{f(t)}{t-z}\,\mathrm{d}t,$$

it is tempting to differentiate with respect to z under the integral sign to find

$$f'(z) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma} \frac{f(t)}{(t-z)^2}\,\mathrm{d}t$$

and the justification of this, via $(f(z+h) - f(z))/h$, is not difficult. But then, we can differentiate again (with essentially the same justification), to find that $f''(z)$ exists and is equal to

$$\frac{2}{2\pi\mathrm{i}} \oint_{\Gamma} \frac{f(t)}{(t-z)^3}\,\mathrm{d}t.$$

and we have effortlessly established that once a complex function of z is differentiable, so is its derivative! Hence, holomorphic functions are infinitely differentiable. The contrast with real analysis is very marked. Indeed, all the interest in complex analysis is focused on the points where functions fail to be holomorphic, known as singularities or singular points.

Furthermore, we have a formula for the derivatives: continuing to differentiate under the integral sign, we have

$$f^{(n)}(z) = \frac{n!}{2\pi\mathrm{i}} \oint_{\Gamma} \frac{f(t)}{(t-z)^{n+1}}\,\mathrm{d}t,$$

this is rarely used *per se*, but it is the key to Taylor's theorem.

Liouville and the maximum principle

A function is called *entire* if it is holomorphic in the whole complex plane (eg z, e^z). Such a function must have a singularity at infinity, because of:

Liouville's theorem. *Any bounded entire function $f(z)$ is constant.*

That is, if $|f(z)| < M$ for some M and all z, then f is a constant (less than M in modulus). The proof is by looking at Cauchy's integral formula for $f'(z)$ and taking Γ to be a large circle; letting the radius of the circle tend to infinity, we have $f'(z) = 0$.

Liouville's theorem leads to a quick proof of the fundamental theorem of algebra, that a polynomial of degree n has all its roots in the complex plane. This is demonstrated by contradiction; having removed all roots that do lie in the complex plane, the remaining polynomial $p(z)$, if not constant, has no zeroes, is unbounded at infinity, and so its modulus is bounded below; hence $1/p(z)$ is entire and bounded above, leading to the contradiction.

There are various generalisations of Liouville's theorem, among which the Phragmen–Lindelof theorems analyse what happens if the modulus of an entire function is bounded in a specified way (eg exponentially) in certain sectors of the complex plane (eg the right-hand half-plane) as $|z| \to \infty$, thereby relaxing the key assumption of Liouville's theorem. Specifically, if $|f(\mathrm{i}y)| < M$ and, for $\mathrm{Re}\, z > 0$, $|f(z)| < K\mathrm{e}^{|z|^\alpha}$ as $|z| \to \infty$ for some $K > 0$ and $0 \le \alpha < 1$, then $|f(z)| < M$ in the right-hand half-plane. Similar results can be found for other sectors.

Slightly more involved than the proof of Liouville's theorem is that of the maximum principle, in the form of:

The maximum modulus theorem. *If $f(z)$ is holomorphic and non-constant on Γ and in its interior D, then the maximum value of $|f(z)|$ in $D \cup \Gamma$ is achieved at a point on Γ, not in D.*

The proof is by contradiction, assuming that $|f(z)|$ has a maximum at some point z in D, and so not on Γ. Using Cauchy's integral formula on a small circle $|t - z| = \epsilon$ to give

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + \epsilon \mathrm{e}^{\mathrm{i}\theta})\, d\theta,$$

if $|f(z + \epsilon \mathrm{e}^{\mathrm{i}\theta_0})| < |f(z)|$ at any point $z + \epsilon \mathrm{e}^{\mathrm{i}\theta_0}$, continuity of $|f|$ and a simple estimate of the right-hand side lead to the contradiction $|f(z)| > |f(z)|$. This result is a restatement of the maximum principle for real-valued solutions of Laplace's equation, as the function $\log |f(z)|$ is harmonic and an increasing function of $|f(z)|$. The modulus of f can of course have a minimum, namely zero, so there is no minimum modulus theorem (unlike for solutions of Laplace's equation).

Taylor

Knowing that a holomorphic function has derivatives of all orders, we expect it to have power series representation. It does:

Taylor's theorem. *If $f(z)$ is holomorphic in a disc $D(a; R)$, then there is a series representation*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

which converges to $f(z)$ for all $0 \le |z - a| < R$. Moreover, $c_n = f^{(n)}(a)/n!$.

Note that the series converges and it converges to $f(z)$; the latter need not be true in real analysis (eg the function e^{-1/x^2} has the Taylor series 0 at the origin, as all its derivatives exist but vanish there).

The circle of convergence for the series is the largest disc within which the series converges, and so the radius of convergence is the distance from $z = a$ to the nearest singular point of $f(z)$. The series diverges for $|z - a| > R$, while on the circle of convergence it may converge at some points (but must diverge at at least one).

One approach to holomorphic functions is to define them as the sums of convergent power series, from which term-by-term differentiation is justified by standard real analysis techniques (such functions are often called analytic). This has the disadvantage that the geometry of the circle of convergence is built in from the start, even though the domain in which $f(z)$ is holomorphic may be another shape altogether. Unless one has a formula for $f(z)$ (eg, $\sum_0^\infty z^n = 1/(1-z)$), to get outside this circle one has to form another series centred at a point near the boundary of the original circle, and hope that the new series converges in a disc containing points outside the original circle of convergence. The process is then repeated with a new circle, and so on. This procedure is called *analytic continuation.*

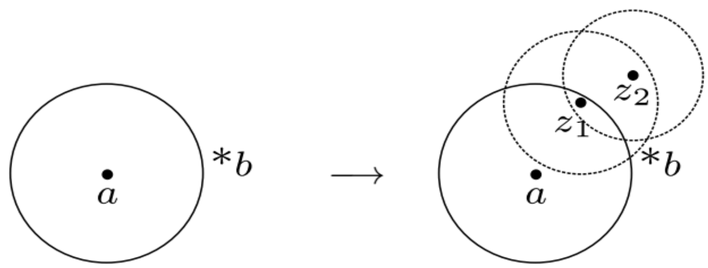

Figure 5: Analytic continuation of $f(z)$ out of its original circle of convergence. The radius of convergence is the distance from $z = a$ to $z = b$, the nearest singular point of $f(z)$.

It should not be thought that analytic continuation is automatically possible. The function

$$f(z) = \sum_{n=0}^{\infty} z^{n!}$$

is holomorphic for $|z| < 1$, by comparison with the geometric series, but the sum diverges at all points $z = \mathrm{e}^{\mathrm{i}\theta}$ for which θ is a rational multiple of 2π (to see this, take $\theta = 2\pi p/q$, and then for all $n \geq q$, $(\mathrm{e}^{\mathrm{i}\theta})^{n!} = 1$). The unit circle is said to be a *natural boundary* for this function.

The guts of the proof of Taylor's theorem are in the following calculation. Taking $a = 0$ without loss of generality, and Γ to be a circle $|z| < |t| < R$,

$$\begin{aligned} f(z) &= \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \frac{f(t)}{t-z}\,\mathrm{d}t \\ &= \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \frac{f(t)}{t(1-z/t)}\,\mathrm{d}t \\ &= \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \sum_{n=0}^{\infty} \frac{z^n}{t^{n+1}}\,\mathrm{d}t \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n; \end{aligned}$$

we have used uniform convergence and Cauchy's integral formula for derivatives along the way.

The identity theorem (isolated zeroes)

One may ask how much information is needed to specify a holomorphic function uniquely. By Taylor's theorem, knowledge of all the derivatives at a point is enough. Another sufficient set of information is given by the following:

Identity theorem. *Suppose $f_1(z)$ and $f_2(z)$ are both holomorphic in D. If there is a sequence of points $z_n \in D$, having has an accumulation point which also lies in D, such that $f_1(z_n) = f_2(z_n)$, then $f_1(z) \equiv f_2(z)$ in D.*

An alternative version of the theorem is that if f_1 and f_2 agree on a dense set, they agree everywhere. Note that it is important that the accumulation point is also in D.