Lecture 2: Multiple valued functions (multifunctions)

A function $f(z)$ has a branch point at $z = a$ if, on taking a circuit round a, the final value of $f(z)$ is not equal to the original one. Examples are $f(z) = z^{\frac{1}{2}}$, where a circuit round the origin takes us from one branch of the square root to the other (from the 'plus' root to the 'minus' root or vice versa), or $f(z) = \log z$, for which an anticlockwise circuit around the origin leaves the real part unchanged but increases the imaginary part by 2π (its multivaluedness stems from the ambiguity in the definition of $\arg z$). These functions also both have branch points at infinity (infinity is a branch point for $f(z)$ if the origin is a branch point for $f(1/z)$).

There are two solutions to this difficulty. One is to extend the domain of definition of the function by constructing its Riemann surface, on which the function is single-valued and holomorphic everywhere except at the branch points (and any other singularities). For example, the Riemann surface for $z^{\frac{1}{2}}$ consists of two copies of the complex plane ('sheets') joined together at the origin and at infinity, and passing through each other in such a way that a complete circuit of the origin takes us from one sheet to the other. The Riemann surface for $\log z$ is like a multistory carpark.

The second solution is to restrict the domain of definition of the function so that the problematic circuits are forbidden. This is achieved by introducing *branch cuts*, joining the branch points, across which contours may not pass. Then it is possible to define single-valued *branches* of the (multi)function, which is regarded as the collection of these branches. For example, we can make $z^{\frac{1}{2}}$ single-valued by putting a cut along the negative real axis, and defining the two branches to be $r^{\frac{1}{2}}e^{i\theta/2}$ and $-r^{\frac{1}{2}}e^{i\theta/2}$, where $r = |z|$ and $\theta = \arg z$ is restricted so that $-\pi < \theta \leq \pi$. There is no need to take the cut along the negative real axis; any curve joining 0 to ∞ will do, and the choice is problem-dependent. With the cut again along the negative real axis and the same restriction on θ, the set of branches of $\log z$ is $\{\log r + i\theta + 2k\pi i\}$, $k \in \mathbb{Z}$; the branch with this cut and $k = 0$ is sometimes called the *principal branch*, written $\operatorname{Log} z$; the corresponding branch of $\arg z$, which is θ above, is written $\operatorname{Arg} z$.

Evaluation of integrals

There is a collection of standard contours which are used to evaluate standard integrals. Sometimes the original integral can be transformed into an integral round a closed contour, but in other cases the contour must be made into a closed one by addition of a suitable return path; the integral along this must be estimated and shown to vanish in a suitable limit. (Sometimes Jordan's result, that $\int_0^\pi e^{-R\sin\theta}\,d\theta \to 0$ as $R \to \infty$, must be invoked; this is a special case of Watson's lemma from asymptotic analysis.) The prominent kinds of integrals are illustrated by the following examples

Example 1

Integrals of rational functions of $\sin\theta$, $\cos\theta$, such as

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta}, \qquad 0 < b < a,$$

for which the substitution $e^{i\theta} = z$ results in the integral of a rational function of z round the unit circle.

Example 2

Integrals of rational functions of x, of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx,$$

where $P(x)$ and $Q(x)$ are polynomials with $\deg(Q) \geq \deg(P) + 2$ and where Q has no real roots; here we first take the integral from $x = -R$ to $x = R$ and close with a large semicircle in the upper half-plane, then let $R \to \infty$, thereby picking up the residues in the upper half-plane. (Closing the contour in the lower half-plane would give the same answer, but we would have to remember an extra minus sign because Γ is then taken clockwise.)

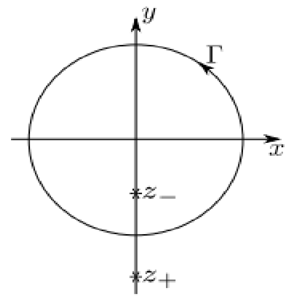

Figure 8: The integrand of the integral after substituting $e^{i\theta} = z$ has two simple poles at $z_\pm$.

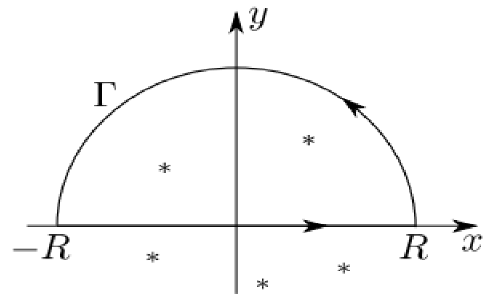

Figure 9: Closing the contour in the upper half-plane.

Example 3

Integrals of the form

$$\int_{-\infty}^{\infty} \frac{P(x)\cos x}{Q(x)}\,\mathrm{d}x,$$

(the cosine may also be a sine), for which we integrate

$$\frac{\mathrm{e}^{\mathrm{i}z}P(z)}{Q(z)}$$

round a semicircular contour closed in the *upper* half-plane; we do this because

$$\mathrm{e}^{\mathrm{i}z} = \mathrm{e}^{\mathrm{i}x-y}$$

is bounded as $y \to +\infty$. Note that it does not work to replace $\mathrm{e}^{\mathrm{i}z}$ by $\cos z$, because the latter is unbounded at infinity (indeed, has an essential singularity there). If the degree of $P(x)$ is one fewer than that of $Q(x)$, the integral may still exist (by cancellation due to the oscillations in the cosine) and the Jordan inequality is necessary to estimate the contribution from the semicircle.

Example 4

Sometimes it is possible to construct a contour Γ such that the integral along one segment of Γ is a constant multiple of the integral along another (together with vanishing contributions from the remaining segments). For example, to evaluate

$$\int_0^{\infty} \frac{\mathrm{d}x}{1+x^{2n}},$$

take Γ to run from 0 or R along the real axis, then via a circular arc to return along the ray $\arg z = \pi/n$ (on which $1 + z^{2n}$ is the same as on the real axis) picking up the residue from the pole at $z = \mathrm{e}^{\mathrm{i}\pi/2n}$. Or, to evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{\cosh x}\,\mathrm{d}x,$$

integrate $\cos z/\cosh z$ round a rectangular contour with corners at $\pm R$, $\pm R + \mathrm{i}\pi$.

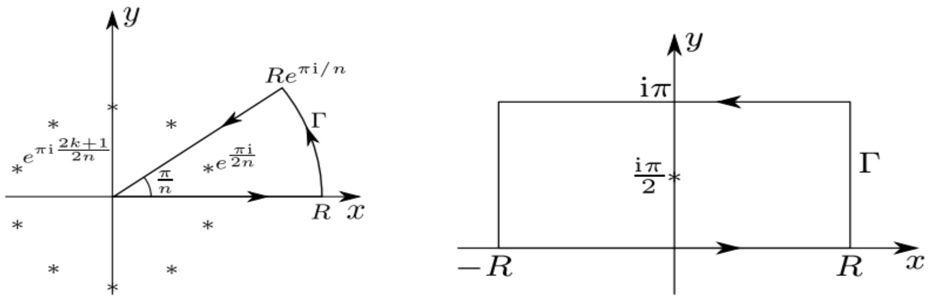

Figure 10: Closed contours for $f(z) = 1/(1 + z^{2n})$ (left) and $f(z) = \cos(z)/\cosh(z)$ (right).

Example 5

Integrals such as

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x$$

can be evaluated by integrating $\mathrm{e}^{\mathrm{i}z}/z$ round a contour consisting of the real axis with a small semicircular indentation at the origin, above the real axis, plus a large semicircle. The small semicircle contributes $-\pi\mathrm{i}$ times the residue of this function at the origin: minus because the semicircle is taken clockwise, and only $\pi\mathrm{i}$ instead of $2\pi\mathrm{i}$ because, having only half a circle, we only pick up half the residue.

Example 6

The 'keyhole' contour is useful for integrals such as

$$\int_0^\infty \frac{\log x\,\mathrm{d}x}{(x+a)(x+b)} \quad (a, b > 0) \qquad \text{or} \qquad \int_0^\infty \frac{\log^2 x\,\mathrm{d}x}{1+x^2};$$

for the first, integrating $(\log z - \pi\mathrm{i})^2/(z+a)(z+b)$ (with the cut along the positive real axis and the choice of the branch $\log z = \log|z| + i\arg z$, with $0 \le \arg z < 2\pi$) round the contour as in Figure 11 and exploiting the different values of the log on either side of the cut along the positive real axis yields the answer; the second is also dealt with using this contour with the function $(\log z - \pi\mathrm{i})^3/(1+z^2)$.

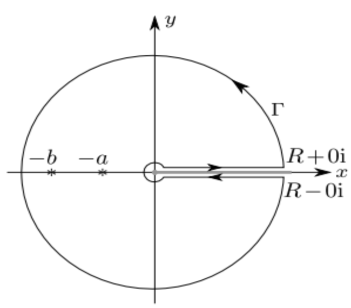

Figure 11: Closed contours for $f(z) = (\log z - \pi\mathrm{i})^2/(z+a)(z+b)$. Note that $\log z - \pi\mathrm{i} = \log(x) \mp \pi\mathrm{i}$ for $z = x \pm 0\mathrm{i}$, $x > 0$.

Example 7

Another example where a multi-function is useful is

$$\int_{-1}^1 \frac{\sqrt{1-x^2}\,\mathrm{d}x}{1+x^2},$$

where we take the cut from -1 to 1 and integrate round it. The contour can be deformed to a large circle at infinity plus two small ones round $z = \pm\mathrm{i}$ as in Figure 12, and the integral evaluated by adding the three corresponding residues. (Care is needed in evaluating the square root at the poles, to get the right branch.)

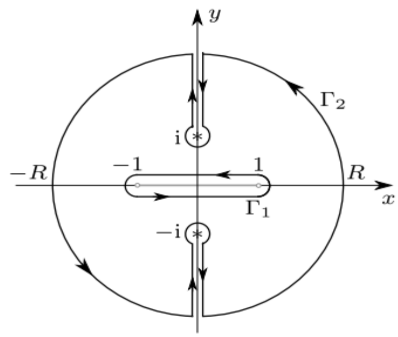

Figure 12: The initial contour Γ_1 can be deformed to Γ_2. The integrals over $z = \pm 0 + \mathrm{i}y$, $|y| > 1$ cancel each other out.

Fourier and Laplace transforms

The Fourier transform of a real function $f(x)$ is

$$\bar{f}(k) \equiv F[f] = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x,$$

and the inverse is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(k)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k$$

(note the asymmetric position of the factor $1/2\pi$; not all authors put it here, so watch out for variations). Inversion is usually accomplished by contour integration in the k-plane.

Integration by parts shows that

$$F\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right] = -\mathrm{i}k\hat{f}(k),$$

and differentiation under the integral sign leads to

$$F[xf(x)] = -\mathrm{i}\frac{\mathrm{d}\bar{f}}{\mathrm{d}k}.$$

The Laplace transform operates on functions defined on the positive real axis:

$$\hat{f}(p) \equiv L[f] = \int_{0}^{\infty} f(x)\mathrm{e}^{-px}\,\mathrm{d}x,$$

and if $f(x)\mathrm{e}^{-\gamma x}$ is integrable (so that $|f(x)|$ grows no worse than $\mathrm{e}^{\gamma x}$ as $x \to \infty$), then $\hat{f}(p)$ exists for $\mathrm{Re}\, p \geq \gamma$ and is holomorphic in p for $\mathrm{Re}\, p > \gamma$; it can usually (being given by a formula) be analytically continued into the rest of the complex p-plane, although singularities inevitably occur. The inversion formula is

$$f(x) = \frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \hat{f}(p)\mathrm{e}^{px}\,\mathrm{d}p.$$

The contour is usually (but not always) completed in the left-hand half-plane and in many problems the solution is given by a sum of residues from the interior of the completed contour, although sometimes a branch cut is also present in $\hat{f}(p)$ and the solution is reduced to an integral along this cut.