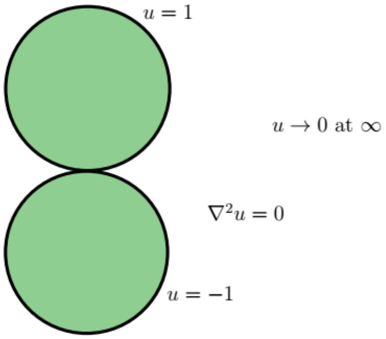
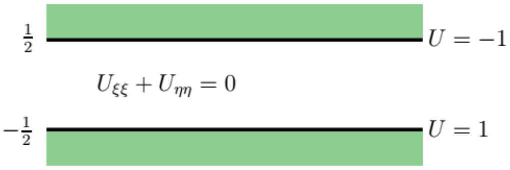
Lecture 5: Solving Laplace's equation by conformal maps



Solution The map $\zeta = 1/z$ takes D onto the strip

$$-\frac{1}{2} < Im(\zeta) < \frac{1}{2}$$

and so we have



with solution

$$U = -2\eta = 2\Re(\mathrm{i}\zeta) = 2\Re(\mathrm{i}/z).$$

Hence

$$u = 2\Re(i/z) = \frac{2y}{x^2 + y^2}.$$

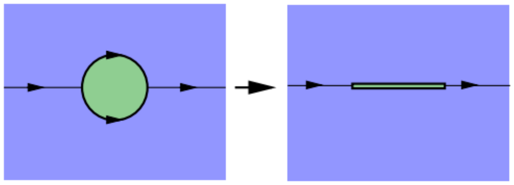
Note that this is bounded in all of D.

Example: Calculate the complex potential for flow past the unit circle with constant velocity $(U_\infty, 0)$ at ∞.

Solution The complex potential is $w = \phi + \mathrm{i}\psi$ and we can take $\psi = 0$ on the x-axis and unit circle. Also $w \sim U_\infty z$ at ∞. Under the Jowkowski map

$$\zeta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

the exterior of the unit circle maps to the ζ-plane cut from -1 to 1:



We need a function $W(\zeta)$ which is real on the ξ axis and at ∞ looks like $U_\infty z \sim 2U_\infty \zeta$. The solution is just

$$W(\zeta) = 2U_\infty \zeta = U_\infty \left(z + \frac{1}{z}\right).$$

NB we can now do Jowkowski by mapping to a slit via a circle.

Example: Find the complex potential for flow over a step of height 1, from $y = 1$, $x < 0$ to $y = 0$, $x > 0$, with velocity $(U_\infty, 0)$ at ∞.

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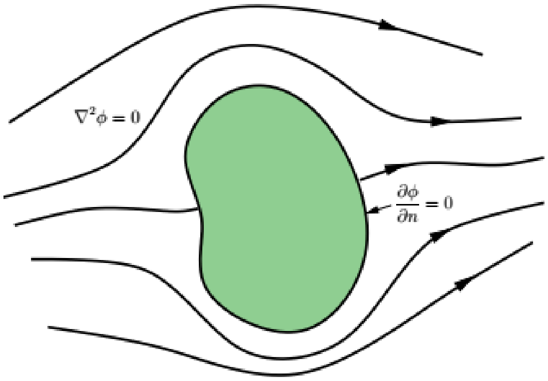
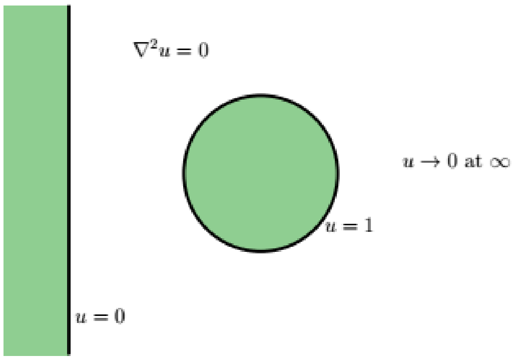
Many applications lead to boundary value problems for Laplace's equation in two dimensions. For example, ideal fluid flow (incompressible, irrotational, inviscid) past a solid obstacle. The obstacle is a streamline on which

$$\frac{\mathrm{d}\phi}{\mathrm{d}n} = 0 \quad \text{so} \quad \psi = \text{constant},$$

where ϕ is the velocity potential and ψ is the streamfunction.



Here both ϕ and ψ (harmonic conjugates) satisfy Laplace's equation and we can solve for either. A second example is steady heat flow with temperature u in a medium:



The idea is to write u (or ϕ or ψ) as the real or imaginary part of a holomorphic function $w(z) = u + \mathrm{i}v$ and then map D onto a simpler domain $f(D)$ by a conformal map

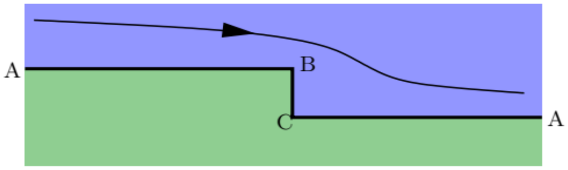
$$\zeta = f(z),$$

in the hope that we can find

$$W(\zeta) = U + \mathrm{i}V = w(z(\zeta))$$

easily. Because a composition of holomorphic functions is holomorphic, $W(\zeta)$ is holomorphic (and its real and imaginary parts satisfy Laplace's equation in $f(D)$).

Example: Find the temperature u in a domain D exterior to the circles $|z - \mathrm{i}| = 1$, $|z + \mathrm{i}| = 1$ with $u = \pm 1$ on $|z \mp \mathrm{i}| = 1$ and $u \to 0$ at ∞.



Solution We map the half-plane $Im(Z) > 0$ onto D by Schwarz-Christoffel (using Z not ζ because the roles are reversed). We have

$$\alpha_A = -1, \quad \beta_A = 2, \qquad \alpha_B = \frac{3}{2}, \quad \beta_B = -\frac{1}{2}, \qquad \alpha_C = \frac{1}{2}, \quad \beta_B = \frac{1}{2}.$$

Let us map

$$Z = -1 \text{ to } B, \qquad Z = +1 \text{ to } C, \qquad Z = \infty \text{ to } A.$$

Then

$$\frac{dz}{dZ} = C\left(\frac{Z+1}{Z-1}\right)^{1/2}$$

from which

$$z = A + C\int^Z \left(\frac{t+1}{t-1}\right)^{1/2} \mathrm{d}t = A + C\int^Z \frac{t+1}{(t^2-1)^{1/2}}\,\mathrm{d}t = A + C\left((Z^2-1)^{1/2} + \cosh^{-1} Z\right).$$

When $Z = 1$ we want $z = 0$ so $A = 0$. When $Z = -1$ we want $z = \mathrm{i}$ so $\mathrm{i} = C\cosh^{-1}(-1)$, i.e. $C = 1/\pi$. Hence

$$z = \frac{1}{\pi}\left((Z^2-1)^{1/2} + \cosh^{-1} Z\right).$$

At infinity $z \sim Z/\pi$. Thus

$$w(z) \sim U_\infty z \text{ at } \infty \quad \Rightarrow \quad W(Z) = w(z(Z)) \sim \frac{U_\infty Z}{\pi} \text{ at } \infty.$$

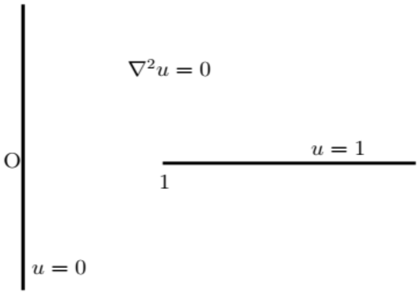
Thus the flow in the Z plane is given by

$$W(Z) = \frac{U_\infty Z}{\pi},$$

so that $w(z)$ is given implicitly by

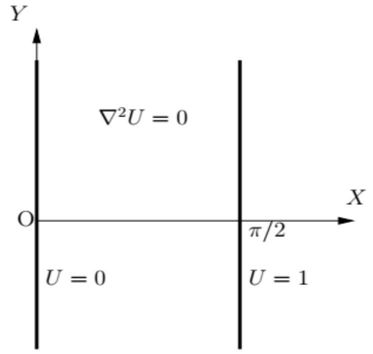
$$z = \frac{1}{\pi}\left(\left(\frac{\pi^2 w^2}{U_\infty^2} - 1\right)^{1/2} + \cosh^{-1}\frac{\pi w}{U_\infty}\right).$$

Example: Solve the BVP ($\sim$ lightning conductor):



with u bounded at ∞.

Solution The domain D is the image of the strip $0 < X < \pi/2$, $-\infty < Y < \infty$ under the map $z = \sin Z$ (see previous examples). In the Z-plane we have

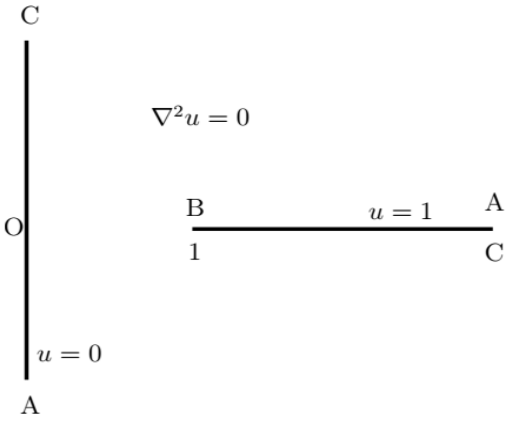


so that $U = \Re(W) = \Re(2Z/\pi)$. Thus

$$u = \Re\left(\frac{2\sin^{-1} z}{\pi}\right).$$

Note that $|\nabla u| \to \infty$ at the tip of the spike.

Alternatively, map the domain to the upper-half Z-plane using the Schwarz-Christoffel formulae:



$$\alpha_A = -\frac{1}{2}, \quad \beta_A = \frac{3}{2}, \qquad \alpha_B = 2, \quad \beta_B = -1, \qquad \alpha_C = -\frac{1}{2}, \quad \beta_C = \frac{3}{2}.$$

Map

$$Z = -1 \text{ to } A, \qquad Z = 0 \text{ to } B, \qquad Z = 1 \text{ to } C.$$

Because of symmetry we are also free to map $Z = \infty$ to $z = 0$. Then

$$\frac{\mathrm{d}z}{\mathrm{d}Z} = \frac{CZ}{(1 - Z^2)^{3/2}}.$$

Thus

$$z = A - \frac{C}{(1 - Z^2)^{1/2}}.$$

$$Z = \infty \text{ when } z = 0 \quad \Rightarrow \quad A = 0.$$

$$Z = 0 \text{ when } z = 1 \quad \Rightarrow \quad C = -1.$$

Thus

$$z = \frac{1}{\sqrt{1 - Z^2}}.$$

The problem in the Z plane is

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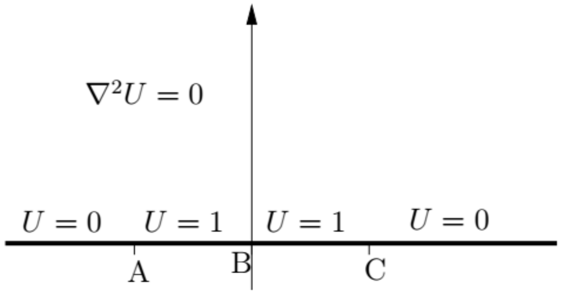
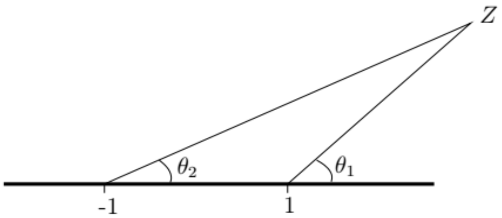


The solution bounded at $Z = \pm 1$ is simply

$$U = \frac{1}{\pi}(\theta_1 - \theta_2) = Im\left(\frac{1}{\pi}\log\left(\frac{Z-1}{Z+1}\right)\right),$$

where θ_1 and θ_1 are the polar angles shown.



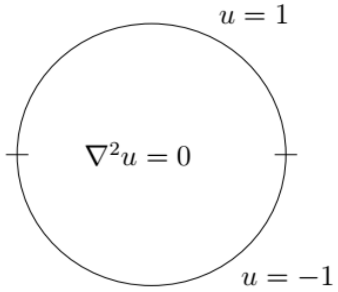
Hence,

$$u = \frac{1}{\pi} Im \log\left(\frac{\sqrt{z^2-1}-z}{\sqrt{z^2-1}+z}\right),$$

which is the same as above. You can get the solution for U by conformal maps using $\zeta = \log((Z-1)/(Z+1))$, illustrating that conformal maps are not always the best way!

Notes

1 You can get the solution for U by conformal maps using $\zeta = \log((Z-1)/(Z+1))$, illustrating that conformal maps are not always the best way! The same technique works for



using angle in semicircle theorem (exercise).

2 If we didn't require U to be bounded at $Z = \pm 1$ then we'd have "eigensolutions" of the form

$$U = Im\left(\frac{1}{Z-1}\right),\ Im\left(\frac{1}{Z+1}\right),\ Im\left(\frac{1}{(Z-1)^2}\right),\ Im\left(\frac{1}{(Z^2-1)}\right),\ Im\left(\frac{1}{(Z+1)^2}\right),\ \cdots$$

and by Liouville and Schwarz reflection we could prove that these are the only solutions with this degree of singularity at $Z = \pm 1$.

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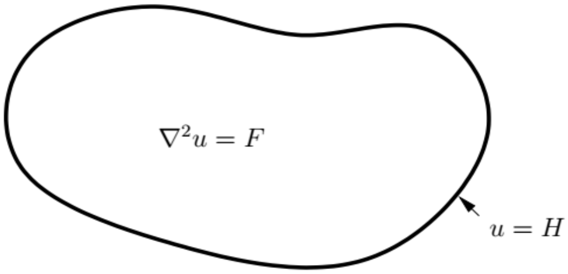
Example: Finding Green's functions



Recall that for Laplace's equation (or Poisson's equation) in a domain D with Dirichlet data the Green's function $G(z, z_0)$ satisfies

$$\nabla^2 G = 0 \quad z \quad \text{on } D\backslash\{z_0\} \qquad (z = x + \mathrm{i}y),$$

with

$$G = 0 \quad \text{on } \mathrm{d}D, \tag{3}$$

$$G \sim \frac{1}{2\pi}\log|z - z_0| \quad \text{as } z \to z_0, \tag{4}$$

and then

$$u(z_0) = \iint_D G(z, z_0)F(z)\,\mathrm{d}x\mathrm{d}y + \int_{\mathrm{d}D} \frac{\mathrm{d}G}{\mathrm{d}n}(z, z_0)H(z)\,ds.$$

We can find G if we know the conformal map $\zeta = f(z)$ taking D onto the unit disc $|\zeta| < 1$. It would be natural to specify that $f(z_0) = 0$, for then the problem for G in the ζ-plane is

$$\nabla^2 G = 0, \qquad 0 < |\zeta| < 1,$$

$$G = 0 \quad \text{on } |\zeta| = 1,$$

$$G \sim \frac{1}{2\pi}\log|\zeta| \quad \text{as } \zeta \to 0,$$

and the unique solution is

$$G = \frac{1}{2\pi}\log|\zeta| = \frac{1}{2\pi}\log|f(z)|.$$

However, this needs a different f for each z_0 and it is better to keep f fixed. Having found this map, $\zeta_1 = f(z)$ say, the singular point will map to some point $\zeta_{10} = f(z_0)$. We can then map the unit ζ_1 disc to itself by

$$\zeta = \frac{\zeta_1 - \zeta_{10}}{1 - \bar{\zeta}_{10}\zeta_1}$$

which takes ζ_{10} to the origin. Then

$$G = \frac{1}{2\pi}\log|\zeta| = \frac{1}{2\pi}\log\left|\frac{f(z) - f(z_0)}{1 - \overline{f(z_0)}f(z)}\right|.$$

Aside: if we write

$$G = \frac{1}{2\pi}\log|z - z_0| + \tilde{G}(z, z_0)$$

then $\tau(z) = \tilde{G}(z, z)$ satisfies Liouville's equation (Ockendon *et al.* p.222)!