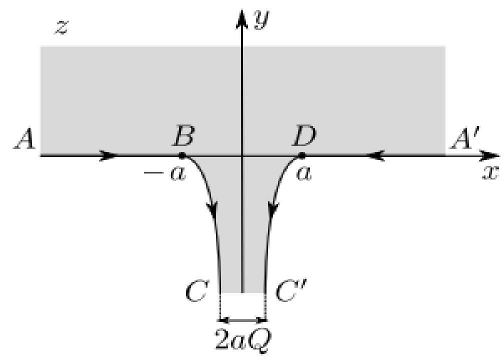
Lecture 7: Flow out of a slot



AB and DA' are walls and the flow leaves tangentially at B, D. Scale z with a and w with $U_\infty a$, where U_∞ is the jet speed at C. If $p \to p_\infty$, $\mathbf{u} \to \mathbf{0}$ as $Im(z) \to \infty$, then

$$p + \frac{1}{2}\rho|\mathbf{u}|^2 = p_\infty,$$

by Bernoulli, so applying this on BC or DC', at C where $\mathbf{u} = (0, -U_\infty)$,

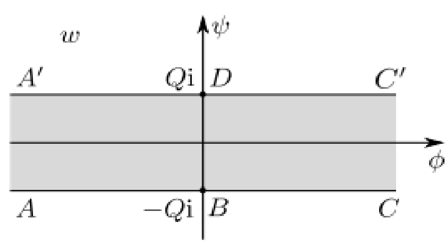
$$\frac{1}{2}\rho U_\infty^2 = p_\infty - p_a,$$

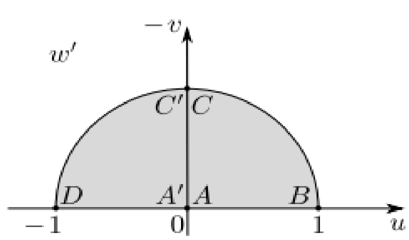
and scaling as above

$$|\mathbf{u}|^2 = \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 = 1$$

on the free surfaces. Also $\psi = -Q$ on ABC and $\psi = Q$ on $C'DA'$ where Q is to be found: it is the contraction ratio $|CC'|/|BD|$.

Step 1 Construct potential and hodograph planes. Note $\phi \sim -U_\infty y$ at C, *i.e.* $\phi \to \infty$ there.





A, A': $u = v = 0$.
B: $u = 1$, $v = 0$, $w' = 1$.
C: $u = 0$, $v = -1$, $w' = \mathrm{i}$.
D: $u = -1$, $v = 0$, $w' = -1$.

Step 2 Map both to an auxillary ζ plane. The map

$$\zeta = -\left(\frac{1 - w'}{1 + w'}\right)^2$$

takes the hodograph plane to the UHP

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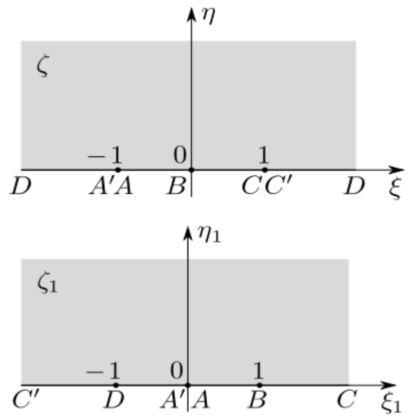

and the map $\zeta_1 = \mathrm{i}\,\mathrm{e}^{\pi w/2Q}$ maps the potential plane to the UHP but with the wrong correspondence of points. Hence put

$$\zeta = \frac{\zeta_1 - 1}{\zeta_1 + 1}$$

to give

$$\left(\frac{1-w'}{1+w'}\right)^2 = \frac{1 - \mathrm{i}\,\mathrm{e}^{\pi w/2Q}}{1 + \mathrm{i}\,\mathrm{e}^{\pi w/2Q}}.$$

This is possible—but complicated—to solve.

If we just want the free surface shape there is a short cut to a parametric form. The key step is to use the natural parametrisation $w' = \mathrm{e}^{-\mathrm{i}\theta}$ on the free surface (θ is the angle the surface makes with the x-axis). *E.g.* on BC $-\pi/2 < \theta < 0$, $0 < \xi < 1$; on CD $-\pi < \theta < -\pi/2$, $1 < \xi < \infty$. Then, on BC,

$$\zeta = -\left(\frac{1 - \mathrm{e}^{-\mathrm{i}\theta}}{1 + \mathrm{e}^{-\mathrm{i}\theta}}\right)^2 = \tan^2\theta/2.$$

We also use

$$\frac{\mathrm{d}z}{\mathrm{d}\zeta} = \frac{1}{w'}\frac{\mathrm{d}w}{\mathrm{d}\zeta} = \frac{\mathrm{d}z}{\mathrm{d}\theta}\Big/\frac{\mathrm{d}\zeta}{\mathrm{d}\theta},$$

so that

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{1}{w'}\frac{\mathrm{d}w}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}\theta}.$$

Then, since

$$\zeta = \frac{\mathrm{i}\mathrm{e}^{\pi w/2Q} - 1}{\mathrm{i}\mathrm{e}^{\pi w/2Q} + 1},$$

we have

$$w = \frac{2Q}{\pi}\log\left(-\mathrm{i}\frac{1+\zeta}{1-\zeta}\right),$$

and therefore

$$\frac{\mathrm{d}w}{\mathrm{d}\zeta} = \frac{2Q}{\pi}\left(\frac{1}{1+\zeta} + \frac{1}{1-\zeta}\right) = \frac{4Q}{\pi(1-\zeta^2)}.$$

Thus

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{1}{w'}\frac{\mathrm{d}w}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}\theta} = e^{\mathrm{i}\theta}\cdot\frac{4Q}{\pi(1-\tan^4\theta/2)}\cdot\sec^2\frac{\theta}{2}\tan\frac{\theta}{2} \tag{5}$$

$$= \frac{4Q}{\pi}e^{\mathrm{i}\theta}\frac{\tan\theta/2}{1-\tan^2\theta/2} = \frac{4Q}{\pi}e^{\mathrm{i}\theta}\frac{\tan\theta/2\,\cos^2\theta^2}{\cos^2\theta/2 - \sin^2\theta/2} \tag{6}$$

$$= \frac{2Q}{\pi}e^{\mathrm{i}\theta}\frac{\sin\theta}{\cos\theta}. \tag{7}$$

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Thus

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} + \mathrm{i}\frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{2Q}{\pi}\tan\theta\,(\cos\theta + \mathrm{i}\sin\theta)\,.$$

Equating real and imaginary parts gives

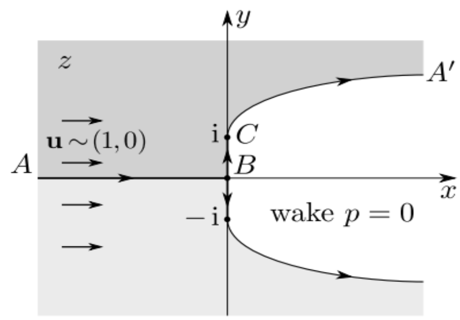
$$x = -1 + \frac{2Q}{\pi}\int_0^\theta \sin t\,\mathrm{d}t = -1 + \frac{2Q}{\pi}(1-\cos\theta), \qquad y = \frac{2Q}{\pi}\int_0^\theta \frac{\sin^2 t}{\cos t}\,\mathrm{d}t,$$

on BC (*i.e.* for $0 \le \theta < \pi/2$). In particular $x(-\pi/2) = -Q = -1 + 2Q/\pi$, giving

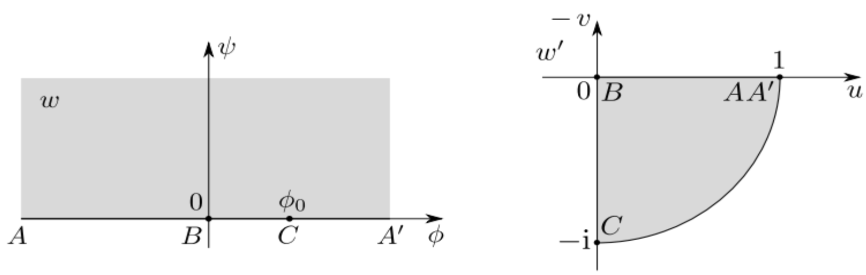
$$Q = \frac{\pi}{\pi+2},$$

which agrees with experiment.

Example: Separating flow past a plate



Step 1 Construct potential and hodograph planes (for flow in $y \ge 0$ using symmetry in x axis, with $\phi_0 > 0$ to be determined):

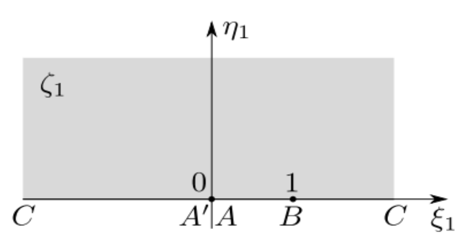


Step 2 Map both to an auxillary ζ plane.

Potential plane is already the upper-half plane. The map

$$\zeta_1 = \left(\frac{(w')^2-1}{(w')^2+1}\right)^2$$

takes the quarter-circle to the UHP, but maps A to 0, B to 1 and C to ∞.



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Hence we want

$$\zeta = \phi_0 \frac{(\zeta_1 - 1)}{\zeta_1} = \phi_0 \left[1 - \left(\frac{(w')^2 + 1}{(w')^2 - 1} \right)^2 \right].$$

Thus

$$w = \phi_0 \left[1 - \left(\frac{(w')^2 + 1}{(w')^2 - 1} \right)^2 \right].$$

To find the equation for the free surface, put $w' = e^{-i\theta}$, giving $w = \phi_0(1 + \cot^2\theta) = \phi_0 \mathrm{cosec}^2\theta$.

$$\frac{dz}{d\theta} = \frac{1}{w'}\frac{dw}{d\theta} = -e^{i\theta} \cdot 2\phi_0 \mathrm{cosec}^2\theta \cot\theta$$

viz.

$$\frac{dx}{d\theta} = -2\phi_0 \frac{\cos^2\theta}{\sin^3\theta}, \qquad \frac{dy}{d\theta} = -2\phi_0 \frac{\cos\theta}{\sin^2\theta}.$$

Integrating

$$x = \phi_0 \cot\theta\, \mathrm{cosec}\theta - \phi_0 \log(\cot\theta + \mathrm{cosec}\theta), \qquad y = \phi_0(\mathrm{cosec}\theta - 1) + 1.$$

We have yet to find ϕ_0. This entails finding $w'(w)$:

$$\frac{w}{\phi_0} = -\frac{4(w')^2}{((w')^2 - 1)^2}$$

so

$$(w')^2 - 1 \pm 2w' i \sqrt{\frac{\phi_0}{w}} = 0,$$

i.e.

$$w' = i \left[\pm \sqrt{\frac{\phi_0}{w}} \pm \sqrt{\frac{\phi_0}{w} - 1} \right].$$

Now, on BC, $w = \phi$ with $0 < \phi < \phi_0$ and w' is negative imaginary. Since the first term is the larger, this means the first sign must be a minus. At B we have $w = 0$, $w' = 0$, which means that the second sign must be plus (opposite to the first). Thus

$$w' = -i \left[\sqrt{\frac{\phi_0}{w}} - \sqrt{\frac{\phi_0}{w} - 1} \right].$$

Thus, on BC, where $w = \phi$ and $z = iy$,

$$\frac{1}{i}\frac{d\phi}{dy} = -i \left[\sqrt{\frac{\phi_0}{\phi}} - \sqrt{\frac{\phi_0}{\phi} - 1} \right].$$

Separating the variables and integrating gives the equation which determines ϕ_0:

$$\int_B^C dy = 1 = \int_0^{\phi_0} \frac{d\phi}{\sqrt{\frac{\phi_0}{\phi}} - \sqrt{\frac{\phi_0}{\phi} - 1}}.$$

Put $\phi = \phi_0 \sin^2\chi$ to give

$$1 = \int_0^{\pi/2} \frac{2\phi_0 \sin\chi\, \cos\chi\, d\chi}{\frac{1}{\sin\chi} - \frac{\cos\chi}{\sin\chi}} = \phi_0 \int_0^{\pi/2} \frac{2\sin^2\chi\, \cos\chi\, d\chi}{1 - \cos\chi} \tag{8}$$

$$= \phi_0 \int_0^{\pi/2} 2\cos\chi\,(1 + \cos\chi)\, d\chi = (2 + \pi/2)\phi_0. \tag{9}$$

Hence

$$\phi_0 = \frac{2}{\pi + 4}$$

At infinity, $\theta \to 0$, $x \to \phi_0/\theta^2$, $y \sim \phi_0/\theta$, viz

$$y \sim \sqrt{\frac{2x}{\pi + 4}}.$$

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