# Lecture 10

**Problem 2:** Suppose we are given $Im(w_\pm) = g_\pm$ on $\Gamma = \{x + \mathrm{i}y \; : \; 0 < x < c, \; y = 0\}$, with $w$ holomorphic on $\mathbb{C}\backslash\overline{\Gamma}$. Find $w$ when (1) $g_+ = -g_- = g$ and (2) $g_+ = g_- = g$, where $g$ is continuous on $\overline{\Gamma}$.

**Remark:** If $w = u + \mathrm{i}v$ then this problem is equivalent to the problem of finding $v$ such that $\nabla^2 v = 0$ away from $\overline{\Gamma}$, and $v_\pm = g_\pm$ on $\Gamma$.

**Solution:** Seek a solution for $w$ of the form

$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_0^c \frac{f(\xi)\,\mathrm{d}\xi}{\xi - z},$$

which is holomorphic on $\mathbb{C}\backslash\overline{\Gamma}$ assuming $f$ to be sufficiently regular. The Plemelj formulae (11)–(12) become

$$w_\pm = u_\pm + \mathrm{i}g_\pm = \pm\frac{1}{2}f + F \quad \text{on} \quad \Gamma, \tag{13}$$

where

$$F(x) = \frac{1}{2\pi \mathrm{i}} ⨍_0^c \frac{f(\xi)}{\xi - x}\mathrm{d}\xi.$$

Note that $F$ is real on $\Gamma$ if and only if $f$ is pure imaginary on $\Gamma$ (because $\xi$, $x$ are real on $\Gamma$).

**Case (1):** If $g_+ = -g_- = g$, (13) becomes

$$w_+ + w_- = u_+ + u_- = 2F \qquad \text{on} \quad \Gamma, \tag{14a}$$

$$w_+ - w_+ = u_+ - u_- + 2\mathrm{i}g = f \qquad \text{on} \quad \Gamma. \tag{14b}$$

By (14a), $F$ must be real, and hence $f$ pure imaginary, on $\Gamma$; thus, by (14b), $u_+ - u_- = 0$ and $f = 2\mathrm{i}g$ on $\Gamma$. It follows that a solution for $w$ is given by

$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_0^c \frac{2\mathrm{i}g(\xi)\,\mathrm{d}\xi}{\xi - z} + h(z),$$

where $h(z)$ is an arbitrary function of $z$ that is holomorphic on $\mathbb{C}\backslash\{0, c\}$ and real on $\Gamma$ ($h$ being trivially a solution of the homogeneous problem in which $g = 0$).

**Case (2):** If $g_+ = g_- = g$, (13) becomes

$$w_+ + w_- = u_+ + u_- + 2\mathrm{i}g = 2F \qquad \text{on} \quad \Gamma, \tag{15a}$$

$$w_+ - w_+ = u_+ - u_- = f \qquad \text{on} \quad \Gamma. \tag{15b}$$

By (15b), $f$ must be real, and hence $F$ pure imaginary, on $\Gamma$; thus, by (15a), $u_+ + u_- = 0$ and $F = \mathrm{i}g$ on $\Gamma$. It follows that

$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_0^c \frac{f(\xi)\,\mathrm{d}\xi}{\xi - z}$$

is a solution provided $f$ satisfies the Cauchy singular integral equation

$$\frac{1}{\pi} ⨍_0^c \frac{f(\xi)\,\mathrm{d}\xi}{\xi - z} = -2g(x) \qquad (0 < x < c), \tag{16}$$

which we need to invert to find $f$.

**Remark:** In case (1) the data gives $w_+ - w_-$ and hence $f$ directly. In case (2) the data gives $w_+ + w_-$ leading to a Cauchy singular integral equation for $f$.

# Lecture 10

**Trick:** Suppose we can find an auxillary function $\tilde{w}(z)$ that is holomorphic and non-zero on $\mathbb{C}\backslash\overline{\Gamma}$ and satisfies $\tilde{w}_+ = -\tilde{w}_- \neq 0$ on $\Gamma$, *i.e.* $\tilde{w}$ is a solution of the homogeneous problem (in which $g = 0$) that is non-zero on $\Gamma$. Let $W = w/\tilde{w}$, then

$$W_+ - W_- = \frac{w_+}{\tilde{w}_+} - \frac{w_-}{\tilde{w}_-} = \frac{w_+}{\tilde{w}_+} - \frac{w_-}{(-\tilde{w}_+)} = \frac{w_+ + w_-}{\tilde{w}_+} = \frac{2\mathrm{i}g}{\tilde{w}_+} \qquad \text{on} \quad \Gamma.$$

If $\tilde{w}_+$ is known, then $W_+ - W_-$ is known (because $g$ is known), so we can turn a problem in which $\bullet_+ + \bullet_-$ is given for $w$ into a problem in which $\bullet_+ - \bullet_-$ is given for $w/\tilde{w}$. By problem (1), a solution for $W$ is given by

$$W(z) = \frac{1}{2\pi\mathrm{i}}\int_0^c \frac{\tilde{f}(\xi)\,\mathrm{d}\xi}{\xi - z} + \tilde{H}(z),$$

where $\tilde{f} = 2\mathrm{i}g/\tilde{w}_+$ on $\Gamma$ and $\tilde{H}$ is an arbitrary function of $z$ holomorphic on $\mathbb{C}\backslash\{0, c\}$. Let us check this result: by the Plemelj formulae,

$$W_\pm(x) = \pm\frac{1}{2}\tilde{f}(x) + \frac{1}{2\pi\mathrm{i}}\int_0^c \frac{\tilde{f}(\xi)\,\mathrm{d}\xi}{\xi - x} + \tilde{H}(x) \qquad (0 < x < c),$$

so $\tilde{f} = W_+ - W_- = 2\mathrm{i}g/\tilde{w}_+$ on $\Gamma$, as required. Moreover,

$$W_+ + W_- = \frac{1}{\pi\mathrm{i}}⨍_0^c \frac{\tilde{f}(\xi)\,\mathrm{d}\xi}{\xi - x} + 2\tilde{H}(x).$$

Since $\tilde{w}_- = -\tilde{w}_+ \neq 0$ and $w_+ - w_- = f$ on $\Gamma$, we also have that

$$W_+ + W_- = \frac{w_+}{\tilde{w}_+} + \frac{w_-}{\tilde{w}_-} = \frac{w_+}{\tilde{w}_+} + \frac{w_-}{(-\tilde{w}_+)} = \frac{w_+ - w_-}{\tilde{w}_+} = \frac{f}{\tilde{w}_+} \qquad \text{on} \quad \Gamma.$$

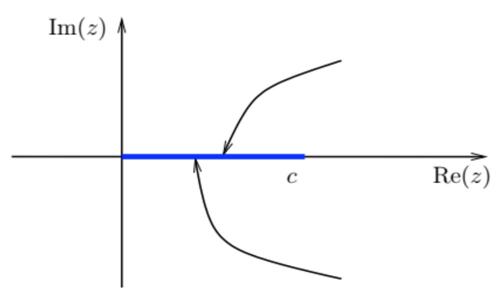
Since $\tilde{f} = 2\mathrm{i}g/\tilde{w}_+$ on $\Gamma$, we deduce that

$$f(x) = \tilde{w}_+(x)\left(W_+(x) + W_-(x)\right) = 2\tilde{w}_+(x)\left(\frac{1}{\pi}⨍_0^c \frac{g(\xi)\,\mathrm{d}\xi}{\tilde{w}_+(\xi)(\xi - x)} + \tilde{H}(x)\right)$$

satisfies the Cauchy singular integral equation (16).

**Finding $\tilde{w}$**

We are left with finding an auxillary function $\tilde{w}(z)$ that is holomorphic on $\mathbb{C}\backslash\overline{\Gamma}$ and satisfies $\tilde{w}_+ = -\tilde{w}_- \neq 0$ on $\Gamma$, where $\Gamma = \{x + \mathrm{i}y \;:\; 0 < x < c,\; y = 0\}$ and $\overline{\Gamma} = \{x + \mathrm{i}y \;:\; 0 \leq x \leq c,\; y = 0\}$. To find $\tilde{w}$, we need to find a function whose value as $\Gamma$ is approached from above is minus that as $\Gamma$ is approached from below.




**Example (1):** When $c = \infty$, we can use $\tilde{w}(z) = z^{1/2}$ provided we take the branch cut along the positive real axis, *e.g.* $z^{1/2} = r^{1/2}\mathrm{e}^{\mathrm{i}\theta/2}$ for $z = r\mathrm{e}^{\mathrm{i}\theta}$ $(r > 0,\, 0 < \theta \leq 2\pi)$, in which case $\tilde{w}_\pm(x) = \pm x^{1/2} \neq 0$ for $x > 0$.

**Example (2):** When $c < \infty$, we can use $\tilde{w}(z) = z^{1/2}(c - z)^{1/2}$, where we take the branch cut along $\Gamma$ and $\tilde{w}_\pm(x) = \pm x^{1/2}(c - x)^{1/2} \neq 0$ for $0 < x < c$.

# Lecture 10

**Remark:** Note that in example (2), we can obtain another valid solution by multiplying $\tilde{w}(z)$ by any function of $z$ that is holomorphic and non-zero on $\mathbb{C}\backslash\{0, c\}$.

What about a general method of finding $\tilde{w}$? We have $\tilde{w}_+/\tilde{w}_- = -1$ on $\Gamma$, so

$$\log \tilde{w}_+ - \log \tilde{w}_- = \log(-1) = (2m+1)\pi\mathrm{i} \qquad \text{on} \quad \Gamma,$$

where $m \in \mathbb{Z}$ (corresponds to the infinite number of branches of the logarithm). From problem (1) in the previous lecture, we deduce that

$$\begin{aligned}\log \tilde{w}(z) &= \frac{1}{2\pi\mathrm{i}} \int_0^c \frac{(2m+1)\pi\mathrm{i}}{\xi - z}\, d\xi + \tilde{h}(z) \\ &= \left(m + \frac{1}{2}\right) \Big[\log(\zeta - z)\Big]_{\zeta=0}^{\zeta=c} + \tilde{h}(z) \\ &= h^*(z) \left(\frac{c-z}{z}\right)^{m+1/2}\end{aligned}$$

where $\tilde{h}(z)$, and hence $h^*(z) = (-1)^m \mathrm{i} e^{\tilde{h}(z)}$, is an arbitrary function of $z$ holomorphic on $\mathbb{C}\backslash\{0, c\}$.

**Important remark concerning case (2):** It is necessary to prescribe the singularities of $w(z)$ at $z = 0$, $c$ and $\infty$ in order to fix $\tilde{H}$ and $\tilde{h}$: often easier to choose $\tilde{h}$ to make the Cauchy integral in $w(z)$ bounded at the endpoints 0, $c$ and then fix the behaviour at 0, $c$ and $\infty$ via $\tilde{H}$. Two concrete examples next time.